GLOBAL SMALL SOLUTIONS TO THREE-DIMENSIONAL INCOMPRESSIBLE MHD SYSTEM

LI XU AND PING ZHANG

Abstract. In this paper, we consider the global wellposedness of 3-D incompressible magneto-hydrodynamical system with small and smooth initial data. The main difficulty of the proof lies in establishing the global in time L^1 estimate for gradient of the velocity field due to the strong degeneracy and anisotropic spectral properties of the linearized system. To achieve this and to avoid the difficulty of propagating anisotropic regularity for the transport equation, we first write our system (1.1) in the Lagrangian formulation (2.20). Then we employ anisotropic Littlewood-Paley analysis to establish the key L^1 in time estimates to the velocity and the gradient of the pressure in the Lagrangian coordinate. With those estimates, we prove the global wellposedness of (2.20) with smooth and small initial data by using the energy method. Toward this, we will have to use the algebraic structure of (2.20) in a rather crucial way. The global wellposedness of the original system (1.1) then follows by a suitable change of variables together with a continuous argument. We should point out that compared with the linearized systems of 2-D MHD equations in [22] and that of the 3-D modified MHD equations in [21], our linearized system (3.1) here is much more degenerate, moreover, the formulation of the initial data for (2.20) is more subtle than that in [22].

Keywords: Inviscid MHD system, Anisotropic Littlewood-Paley theory, Dissipative estimates, Lagrangian coordinates

AMS Subject Classification (2000): 35Q30, 76D03

1. Introduction

In this paper, we investigate the global wellposedness of the following three-dimensional incompressible magnetic hydrodynamical system (or MHD in short) with initial data being sufficiently close to the equilibrium state:

$$
\left\{
\begin{array}{l}
\partial_t \boldsymbol{b} + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \boldsymbol{b}\cdot\nabla\boldsymbol{u}, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\
\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p = -\frac{1}{2}\nabla|\boldsymbol{b}|^2 + \boldsymbol{b}\cdot\nabla\boldsymbol{b},\\
\operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{b} = 0,\\
\boldsymbol{b}|_{t=0} = \boldsymbol{b}_0, \quad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0,
\end{array}
\right.
\tag{1.1}
$$

where $\boldsymbol{b} = (b^1, b^2, b^3)^T$ denotes the magnetic field, and $\boldsymbol{u} = (u^1, u^2, u^3)^T, p$ the velocity and scalar pressure of the fluid respectively. This MHD system (1.1) with zero diffusivity in the equation for the magnetic field can be applied to model plasmas when the plasmas are strongly collisional, or the resistivity due to these collisions are extremely small. One may check the references [13, 17, 4] for more detailed explanations to this system.

It has been a long-standing open problem that whether or not classical solutions of (1.1) can develop finite time singularities even in the two-dimensional case. In the case when there is full magnetic diffusion in (1.1), Duvaut and Lions [14] established the local existence and uniqueness of solution in the classical Sobolev space $H^s(\mathbb{R}^d)$, $s \geq d$, they also proved the

Date: 9/May/2013.

global existence of solutions to this system with small initial data; Sermange and Temam [26] proved the global unique solution in the two space dimensions; Abidi and Paicu [1] proved similar result as in [14] for the so-called inhomogeneous MHD system with initial data in the critical spaces. With mixed partial dissipation and additional magnetic diffusion in the two-dimensional MHD system, Cao and Wu [5] (see also [6]) proved that such a system is globally wellposed for any data in $H^2(\mathbb{R}^2)$. Lin and the second author [21] proved the global wellposedness to a modified three-dimensional MHD system (3-D version of (1.2) below) with initial data sufficiently close to the equilibrium state. Lin and the authors [22] established the global existence of small solutions to the two-dimensional MHD equations (1.1).

For the incompressible MHD equations (1.1), whether there is a dissipation or not for the magnetic field is a very important problem also from physics of plasmas. The heating of high temperature plasmas by MHD waves is one of the most interesting and challenging problems of plasma physics especially when the energy is injected into the system at the length scales much larger than the dissipative ones. It has been conjectured that in the three-dimensional MHD system, energy is dissipated at a rate that is independent of the ohmic resistivity [11]. In other words, the viscosity (diffusivity) for the magnetic field equation can be zero yet the whole system may still be dissipative. We shall justify this conjecture for (1.1) with initial data close enough to the equilibrium state.

Notice that in two space dimensions, $\operatorname{div}\boldsymbol{b}=0$ implies the existence of a scalar function ϕ so that $\boldsymbol{b}=(\partial_2\phi,-\partial_1\phi)^T$, and the system (1.1) becomes

$$
\begin{cases}
\partial_t\phi+\boldsymbol{u}\cdot\nabla\phi=0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\
\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\Delta\boldsymbol{u}+\nabla p=-\frac{1}{2}\nabla|\nabla\phi|^2-\operatorname{div}\big[\nabla\phi\otimes\nabla\phi\big],\\
\operatorname{div}\boldsymbol{u}=0,\\
\phi|_{t=0}=\phi_0(x)=x_2+\psi_0(x), \quad \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0,
\end{cases}
\tag{1.2}
$$

The main idea in [22] is first to seek another scalar function $\widetilde{\phi}(x)=-x_1+\widetilde{\psi}_0$ so that

$$
\det U_0=1 \quad \text{for} \quad U_0=\begin{pmatrix}1+\partial_{x_2}\psi_0 & \partial_{x_2}\tilde{\psi}_0\\ -\partial_{x_1}\psi_0 & 1-\partial_{x_1}\tilde{\psi}_0\end{pmatrix},
\tag{1.3}
$$

provided that ψ_0 is sufficiently small in some sense. Then the authors of [22] looked for a volume preserving diffeomorphism in $\mathbb{R}^2$, $X_0(y)=y+Y_0(y)$, so that

$$
U_0\circ X_0(y)=\nabla_y X_0(y)=I+\nabla_y Y_0(y).
\tag{1.4}
$$

Let $(Y(t,y),q(t,y))$ be determined by

$$
\begin{aligned}
&X(t,y)=X_0(y)+\int_0^t\boldsymbol{u}(s,X(s,y))\,ds\stackrel{\text{def}}{=}y+Y(t,y),\\
&q(t,y)\stackrel{\text{def}}{=}(p+|\nabla\phi|^2)\circ X(t,y).
\end{aligned}
\tag{1.5}
$$

(1.2) can be equivalently reformulated as

$$
\begin{cases}
Y_{tt}-\nabla_Y\cdot\nabla_Y Y_t-\partial_{y_1}^2 Y+\nabla_Y q=\boldsymbol{0}, \qquad (t,y)\in\mathbb{R}^+\times\mathbb{R}^2,\\
\nabla_Y\cdot Y_t=0,\\
Y|_{t=0}=Y_0, \quad Y_t|_{t=0}=\boldsymbol{u}_0\circ X_0(y)\stackrel{\text{def}}{=}Y_1,
\end{cases}
\tag{1.6}
$$

where $\nabla_Y \stackrel{\text{def}}{=} \mathcal{A}_Y^T \nabla_y$ and

$$\mathcal{A}_Y \stackrel{\text{def}}{=} \begin{pmatrix} 1+\partial_{y_2} Y^2 & -\partial_{y_2} Y^1 \\ -\partial_{y_1} Y^2 & 1+\partial_{y_1} Y^1 \end{pmatrix}. \tag{1.7}$$

In particular, the linearized system of (1.6) reads

$$\begin{cases} Y_{tt} - \Delta_y Y_t - \partial_{y_1}^2 Y = \boldsymbol{f}(Y,q), \qquad (t,y) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \nabla_y \cdot Y = \rho(Y), \\ Y|_{t=0} = Y_0, \quad Y_t|_{t=0} = Y_1. \end{cases} \tag{1.8}$$

By using anisotropic Littlewood-Paley theory, the authors [22] first established the global wellposedness of (1.6) with small and smooth initial data, then they proved the global wellposedness of (1.2) with sufficiently small data $(\psi_0, \boldsymbol{u}_0)$ through a suitable changes of variables.

However, in the three-dimensional case, we can not find such an equivalent formulation of (1.1) as (1.2). Instead, for $\boldsymbol{b}_0 - \boldsymbol{e}_3$ being sufficient small, we can find a $\boldsymbol{\Psi} = (\psi_1, \psi_2, \psi_3)^T$ so that there holds (2.3). Compared with (1.3), (2.3) is a nonlinear system. With this $\boldsymbol{\Psi}$, we can define $\bar{\boldsymbol{b}}_0$ and $\tilde{\boldsymbol{b}}_0$ via (2.4) so that the 3×3 matrix $U_0 \stackrel{\text{def}}{=} (\bar{\boldsymbol{b}}_0, \tilde{\boldsymbol{b}}_0, \boldsymbol{b}_0)$ satisfies

$$\operatorname{div}\bar{\boldsymbol{b}} = \operatorname{div}\tilde{\boldsymbol{b}} = 0, \quad \text{and} \quad \det U_0 = 1. \tag{1.9}$$

With thus obtained U_0, we can find a 3-D volume preserving diffeomorphism $X_0(y) = y + Y_0(y)$, and reformulate (1.1) in the Lagrangian coordinate (2.20) with its linearized system (2.21). We point out that one crucial idea in [22] is to use $\partial_{y_1} Y^1 + \partial_{y_2} Y^2 = \rho(Y)$ to propagate the time dissipative estimate of $\partial_{y_1} Y^1$ to that of $\partial_{y_2} Y^2$. Notice that in the linearized system (2.21), one only has time dissipative estimate for $\partial_{y_3} Y$, and we can not use $\nabla_y \cdot Y = \rho(Y)$ to propagate the time dissipative estimate from $\partial_{y_3} Y^3$ to that of $\partial_{y_1} Y^1, \partial_{y_2} Y^2$, which gives rise to another difficulty in the analysis of three-dimensional MHD system. And we will have to use the nonlinear structure of (2.20) in a rather crucial way so that the source term in (2.21) is still globally integrable in time. As in [22], we shall first establish the global wellposedness of (2.20) with small and smooth initial data, we then prove the global existence of small solution to (1.1) by a suitable changes of variables along with a continuous argument.

We should remark that the system (1.2) is of interest not only because it models the incompressible MHD equations, but also because it arises in many other important applications. Moreover, its nonlinear coupling structure is universal, see the recent survey article [19]. Indeed, the system (1.2) resembles the 2-D viscoelastic fluid system:

$$\begin{cases} U_t + \boldsymbol{u} \cdot \nabla U = \nabla \boldsymbol{u} U, \\ \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \Delta \boldsymbol{u} + \nabla \cdot (U U^T), \\ \operatorname{div} \boldsymbol{u} = 0, \\ U|_{t=0} = U_0, \qquad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \end{cases} \tag{1.10}$$

where U denotes the deformation tensor, $\boldsymbol{u}$ is the fluid velocity and p represents the hydrodynamic pressure (we refer to [20] and the references therein for more details).

In two space dimensions, when $\nabla \cdot U_0 = 0$, it follows from (1.10) that $\nabla \cdot U(t,x) = 0$ for all $t > 0$. Therefore, one can find a vector $\boldsymbol{\phi} = (\phi_1, \phi_2)^T$ such that

$$U = \begin{pmatrix} -\partial_2 \phi_1 & -\partial_2 \phi_2 \\ \partial_1 \phi_1 & \partial_1 \phi_2 \end{pmatrix}.$$

Then (1.10) can be equivalently reformulated as

$$\begin{cases} \boldsymbol{\phi}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{\phi} = \boldsymbol{0}, \\ \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \Delta\boldsymbol{u} - \sum_{i=1}^2 \operatorname{div}\left[\nabla\phi_i\otimes\nabla\phi_i\right], \\ \operatorname{div}\boldsymbol{u} = 0, \\ \boldsymbol{\phi}|_{t=0} = \boldsymbol{\phi}_0, \qquad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0. \end{cases} \tag{1.11}$$

The authors ([20]) established the global existence of smooth solutions to the Cauchy problem in the entire space or on a periodic domain for (1.11) in general space dimensions provided that the initial data is sufficiently close to the equilibrium state (one may check [10, 18] for the 3-D result). One sees the only difference between (1.2) and (1.11) lying in the fact that ϕ is a scalar function in (1.2), while $\boldsymbol{\phi} = (\phi_1, \phi_2)^T$ is a vector-valued function with the unit Jacobian in (1.11). However, it gives rise to an essential difficulty in the analysis. In fact, there is a damping mechanism of the system (1.11) that can be seen from the linearization of the system ∂_t (1.11):

$$\begin{cases} \boldsymbol{\phi}_{tt} - \Delta\boldsymbol{\phi} - \Delta\boldsymbol{\phi}_t + \nabla q = \boldsymbol{f}, \\ \boldsymbol{u}_{tt} - \Delta\boldsymbol{u} - \Delta\boldsymbol{u}_t + \nabla p = \boldsymbol{F}, \\ \operatorname{div}\boldsymbol{u} = 0. \end{cases} \tag{1.12}$$

We also remark that the linearized system of (1.2) in 3-D reads

$$\partial_t^2\psi - (\partial_{x_1}^2 + \partial_{x_2}^2)\psi - \Delta\partial_t\psi = f. \tag{1.13}$$

One may check Remark 1.4 of [21] for details. It is easy to observe that our linearized system in (2.21) is much more degenerate than (1.12) and (1.13).

As in [22], to describe the initial data $\boldsymbol{b}_0$ in (1.1), we need the following definition:

Definition 1.1. *Let $\boldsymbol{b}_0 = (b_0^1, b_0^2, b_0^3)^T$ be a smooth enough vector field. We define its trajectory $X(t,x)$ by*

$$\begin{cases} \dfrac{dX(t,x)}{dt} = \boldsymbol{b}_0(X(t,x)), \\ X(t,x)|_{t=0} = x. \end{cases} \tag{1.14}$$

We call that f and $\boldsymbol{b}_0$ are admissible on a domain D of $\mathbb{R}^3$ if there holds

$$\int_{\mathbb{R}} f(X(t,x))\,dt = 0 \quad \text{for} \quad \text{all} \quad x\in D.$$

Remark 1.1. *As in [22], the condition that f and $\boldsymbol{b}$ are admissible on some set of $\mathbb{R}^3$ is to guarantee that*

$$b_0^1\partial_{x_1}\psi + b_0^2\partial_{x_2}\psi + b_0^3\partial_{x_3}\psi = f \tag{1.15}$$

has a solution ψ so that $\lim_{|x|\to\infty}\psi(x) = 0$. Let us take $\boldsymbol{b} = (0,0,1)^T$ for example. In this case, (1.15) becomes $\partial_{x_3}\psi = f$, which together with the condition $\lim_{|x_3|\to\infty}\psi(x) = 0$ ensures that

$$\psi(x_h, x_3) = -\int_{x_3}^{\infty} f(x_h, t)\,dt = \int_{-\infty}^{x_3} f(x_h,t)\,dt.$$

We thus obtain that $\int_{\mathbb{R}} f(x_h,t)\,dt = 0$, that is, f and $(0,0,1)^T$ are admissible on $\mathbb{R}^2\times\{0\}$.

Notations: Let X_1, X_2 be Banach spaces, the norms $\|\cdot\|_{X_1\cap X_2} \overset{\text{def}}{=} \|\cdot\|_{X_1} + \|\cdot\|_{X_2}$ and $\|\cdot\|_{L^p(\mathbb{R}^+;X_1\cap X_2)} \overset{\text{def}}{=} \|\cdot\|_{L^p(\mathbb{R}^+;X_1)} + \|\cdot\|_{L^p(\mathbb{R}^+;X_2)}$ for $p\in[1,\infty]$.

We now state the main result of this paper:

Theorem 1.1. *Let $s_1 > \frac{5}{4}$, $s_2 \in (-\frac{1}{2}, -\frac{1}{4})$, and $p \in (\frac{3}{2}, 2)$. Let $s \geq s_1 + 2$, let $(\boldsymbol{b}_0, \boldsymbol{u}_0)$ satisfy $\boldsymbol{b}_0 - \boldsymbol{e}_3 \in B^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1} \cap H^s(\mathbb{R}^3)$ for $\boldsymbol{e}_3 = (0,0,1)^T$, and $\boldsymbol{u}_0 \in \dot{H}^{s_2} \cap \dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)$ with $\nabla \boldsymbol{u}_0 \in B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1} \cap H^{s-1}(\mathbb{R}^3)$ and*

$$\|\boldsymbol{b}_0 - \boldsymbol{e}_3\|_{B^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1}} + \|\boldsymbol{u}_0\|_{\dot{H}^{s_2} \cap \dot{B}^{\frac{3}{p}-1}_{p,1}} + \|\nabla \boldsymbol{u}_0\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}} \leq c_0 \tag{1.16}$$

for some c_0 sufficiently small. We assume moreover that $\boldsymbol{b}_0 - \boldsymbol{e}_3$ and $\boldsymbol{b}_0$ are admissible on $\mathbb{R}^2 \times \{0\}$ in the sense of Definition 1.1 and $\operatorname{Supp}\big(\boldsymbol{b}_0 - \boldsymbol{e}_3\big)(x_1, x_2, \cdot) \subset [-K, K]$ *for some positive constant K. Then (1.1) has a unique global solution $(\boldsymbol{b}, \boldsymbol{u}, p)$ (up to a constant for p) so that*

$$\begin{aligned}
&\boldsymbol{b} - \boldsymbol{e}_3 \in C([0,\infty); H^s(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1}(\mathbb{R}^3)),\\
&\nabla p \in C([0,\infty); H^{s-1}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^3)),\\
&\boldsymbol{u} \in C([0,\infty); H^s(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}^+; \dot{H}^{s+1}(\mathbb{R}^3)).
\end{aligned} \tag{1.17}$$

Furthermore, there holds

$$\begin{aligned}
&\|\boldsymbol{b} - \boldsymbol{e}_3\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2})} + \|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2})}\\
&\quad + \|\boldsymbol{b} - \boldsymbol{e}_3\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1})} + \|\boldsymbol{u}\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2+1})}\\
&\quad + \|\boldsymbol{u}\|_{L^1(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla p\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2})}\\
&\leq C\big(\|\boldsymbol{b}_0 - \boldsymbol{e}_3\|_{B^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1}} + \|\boldsymbol{u}_0\|_{\dot{H}^{s_2} \cap \dot{B}^{\frac{3}{p}-1}_{p,1}} + \|\nabla \boldsymbol{u}_0\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}}\big).
\end{aligned} \tag{1.18}$$

Remark 1.2. *(1) One may find the definitions of Besov spaces in Subsection 3.2. We remark that those technical assumptions on $\boldsymbol{b}_0$ and $\boldsymbol{u}_0$ will be used to deal with the low frequency part of $\boldsymbol{b}$ and $\boldsymbol{u}$. For simplicity, we do not provide result on the propagation of regularities for $\boldsymbol{b}_0 - \boldsymbol{e}_3 \in B^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1}(\mathbb{R}^3)$ and $\nabla \boldsymbol{u}_0 \in B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}(\mathbb{R}^3)$.*

(2) Here we point out that the estimate of $\|\boldsymbol{b} - \boldsymbol{e}_3\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1})}$ in (1.18) *is not standard for the solutions of the transport equation in* (1.1)*. It is purely due to the coupling structure in* (1.1)*. And this estimate in some sense explains that the magnetic field is indeed time dissipative even without resistivity for the magnetic field. We shall go back to this point in our future work.*

(3) We can improve the condition that: $\operatorname{Supp}\big(\boldsymbol{b}_0 - \boldsymbol{e}_3\big)(x_1, x_2, \cdot) \subset [-K, K]$ *for some positive number K, in Theorem 1.1 by assuming appropriate decay of $\boldsymbol{b}_0 - \boldsymbol{e}_3$ with respect to x_3 variable. For a clear presentation, we prefer not to present this technical part here.*

Let us complete this section by the notation we shall use in this context.

Notation. For any $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^3)$ the classical L^2 based Sobolev spaces with the norm $\|\cdot\|_{H^s}$, while $\dot{H}^s(\mathbb{R}^3)$ the classical homogenous Sobolev spaces with the norm $\|\cdot\|_{\dot{H}^s}$. Let A, B be two operators, we denote $[A; B] = AB - BA$, the commutator between A and B. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$, and $a \sim b$ means that both $a \lesssim b$ and $b \lesssim a$. We shall denote by $(a|b)$ the $L^2(\mathbb{R}^3)$ inner product of a and b. $(d_{j,k})_{j,k\in\mathbb{Z}}$ (resp. $(c_j)_{j\in\mathbb{Z}}$) will be a generic element of $\ell^1(\mathbb{Z}^2)$ (resp. $\ell^2(\mathbb{Z})$) so that $\sum_{j,k\in\mathbb{Z}} d_{j,k} = 1$ (resp. $\sum_{j\in\mathbb{Z}} c_j^2 = 1$). Finally, we denote by $L^p_T(L^q_h(L^r_v))$ the space $L^p([0,T]; L^q(\mathbb{R}^2_{x_h}; L^r(\mathbb{R}_{x_3})))$ with $x_h = (x_1, x_2)$.

2. Lagrangain formulation of (1.1)

Motivated by [22], we are going to construct two vector fields $\bar{\boldsymbol{b}}_0 = (\bar{b}_0^1, \bar{b}_0^2, \bar{b}_0^3)^T$ and $\tilde{\boldsymbol{b}}_0 = (\tilde{b}_0^1, \tilde{b}_0^2, \tilde{b}_0^3)^T$ so that the 3×3 matrix $U_0 \stackrel{\text{def}}{=} (\bar{\boldsymbol{b}}_0, \tilde{\boldsymbol{b}}_0, \boldsymbol{b}_0)$ satisfies (1.9).

Proposition 2.1. *Let $s > 2 + \frac{3}{p}$ and $p \in (\frac{3}{2}, 2)$. Let $\boldsymbol{b}_0 - \boldsymbol{e}_3 = (b_0^1, b_0^2, b_0^3 - 1)^T \in B^s_{p,1}(\mathbb{R}^3)$ with*

$$\operatorname{div} \boldsymbol{b}_0 = 0 \quad \text{and} \quad \|(b_0^1, b_0^2, b_0^3 - 1)\|_{B^s_{p,1}} \leq \varepsilon_0. \tag{2.1}$$

We assume moreover that $\boldsymbol{b}_0 - \boldsymbol{e}_3$ and $\boldsymbol{b}_0$ are admissible on $\mathbb{R}^2 \times \{0\}$ in the sense of Definition 1.1 and Supp $\big(\boldsymbol{b}_0 - \boldsymbol{e}_3\big)(x_1, x_2, \cdot) \subset [-K, K]$ *for some positive constant K. Then for ε_0 sufficiently small, there exists a $\boldsymbol{\Psi} = (\psi_1, \psi_2, \psi_3)^T$ which satisfies*

$$\|(\psi_1, \psi_2, \psi_3\|_{B^s_{p,1}} \leq C(K, \varepsilon_0)\|(b_0^1, b_0^2, b_0^3 - 1)\|_{B^s_{p,1}}, \tag{2.2}$$

and

$$\begin{aligned} &b_0^1 = \partial_{x_2}\psi_1 \partial_{x_3}\psi_2 + \partial_{x_3}\psi_1(1 - \partial_{x_2}\psi_2), \quad b_0^2 = \partial_{x_3}\psi_1 \partial_{x_1}\psi_2 + \partial_{x_3}\psi_2(1 - \partial_{x_1}\psi_1), \\ &b_0^3 = (1 - \partial_{x_1}\psi_1)(1 - \partial_{x_2}\psi_2) - \partial_{x_2}\psi_1 \partial_{x_1}\psi_2, \quad \text{and} \quad \det(I - \nabla_x \boldsymbol{\Psi}) = 1. \end{aligned} \tag{2.3}$$

Moreover, we define

$$\begin{aligned} \bar{\boldsymbol{b}}_0 &\stackrel{\text{def}}{=} \big((1 - \partial_{x_2}\psi_2)(1 - \partial_{x_3}\psi_3) - \partial_{x_3}\psi_2 \partial_{x_2}\psi_3, \partial_{x_3}\psi_2 \partial_{x_1}\psi_3 + \partial_{x_1}\psi_2(1 - \partial_{x_3}\psi_3), \\ &\qquad \partial_{x_1}\psi_2 \partial_{x_2}\psi_3 + \partial_{x_1}\psi_3(1 - \partial_{x_2}\psi_2)\big)^T \quad \text{and} \\ \tilde{\boldsymbol{b}}_0 &\stackrel{\text{def}}{=} \big(\partial_{x_3}\psi_1 \partial_{x_2}\psi_3 + \partial_{x_2}\psi_1(1 - \partial_{x_3}\psi_3), (1 - \partial_{x_1}\psi_1)(1 - \partial_{x_3}\psi_3) - \partial_{x_3}\psi_1 \partial_{x_1}\psi_3, \\ &\qquad \partial_{x_2}\psi_1 \partial_{x_1}\psi_3 + \partial_{x_2}\psi_3(1 - \partial_{x_1}\psi_1)\big)^T, \end{aligned} \tag{2.4}$$

then $U_0 \stackrel{\text{def}}{=} (\bar{\boldsymbol{b}}_0, \tilde{\boldsymbol{b}}_0, \boldsymbol{b}_0)$ satisfies (1.9), and for $\boldsymbol{e}_1 = (1, 0, 0)^T, \boldsymbol{e}_2 = (0, 1, 0)^T$,

$$\|\bar{\boldsymbol{b}}_0 - \boldsymbol{e}_1\|_{B^{s-1}_{p,1}} + \|\tilde{\boldsymbol{b}}_0 - \boldsymbol{e}_2\|_{B^{s-1}_{p,1}} \leq C(K, \varepsilon_0)\|\big(b_0^1, b_0^2, b_0^3 - 1\big)\|_{B^s_{p,1}}. \tag{2.5}$$

The proof of this proposition is postponed in Appendix B.

With U_0 obtained in Proposition 2.1, we shall first investigate the global wellposedness to the following system with sufficiently small u_0 :

$$\begin{cases} \partial_t U + \boldsymbol{u} \cdot \nabla U = \nabla \boldsymbol{u} U, \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = -\dfrac{1}{2}\nabla|\boldsymbol{b}|^2 + \boldsymbol{b} \cdot \nabla \boldsymbol{b}, \\ \operatorname{div} \boldsymbol{u} = 0 \quad \text{and} \qquad \operatorname{div} U = 0, , \\ U|_{t=0} = U_0, \quad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \end{cases} \tag{2.6}$$

where the 3×3 matrix $U = \big(\bar{\boldsymbol{b}}, \tilde{\boldsymbol{b}}, \boldsymbol{b}\big)$, and $\bar{\boldsymbol{b}} = \big(\bar{b}^1, \bar{b}^2, \bar{b}^3\big)^T$, $\tilde{\boldsymbol{b}} = \big(\tilde{b}^1, \tilde{b}^2, \tilde{b}^3\big)^T$. In particular, for any smooth enough solution $(U, \boldsymbol{u})$ of (2.6), $(\boldsymbol{b}, \boldsymbol{u})$ must be a smooth enough solution of (1.1).

The main result concerning the wellposedness of the system (2.6) can be stated as follows:

Theorem 2.1. *Let $s_1 > \frac{5}{4}$, $s_2 \in (-\frac{1}{2}, -\frac{1}{4})$ and $p \in (1, 2)$. Let $\boldsymbol{u}_0 \in \dot{H}^{s_2} \cap \dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)$ with $\nabla \boldsymbol{u}_0 \in B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}(\mathbb{R}^3)$ and $U_0 = \big(I - \nabla_x \boldsymbol{\Psi}\big)^{-1}$ with $\boldsymbol{\Psi} = (\psi_1, \psi_2, \psi_3)^T$ satisfying $\nabla \boldsymbol{\Psi} \in \dot{B}^{s_2+\frac{3}{p}-\frac{3}{2}}_{p,2} \cap B^{s_1+\frac{3}{p}-\frac{1}{2}}_{p,1}(\mathbb{R}^3)$ and $\det(I - \nabla_x \boldsymbol{\Psi}) = 1$. We assume that*

$$\|\nabla \boldsymbol{\Psi}\|_{\dot{B}^{s_2+\frac{3}{p}-\frac{3}{2}}_{p,2} \cap B^{s_1+\frac{3}{p}-\frac{1}{2}}_{p,1}} + \|\boldsymbol{u}_0\|_{\dot{H}^{s_2} \cap \dot{B}^{\frac{3}{p}-1}_{p,1}} + \|\nabla \boldsymbol{u}_0\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}} \leq \varepsilon_0 \tag{2.7}$$

for some ε_0 sufficiently small. Then (2.6) has a unique global solution $(U, \boldsymbol{u}, p)$ (up to a constant for p), with $U = \big(\bar{\boldsymbol{b}}, \tilde{\boldsymbol{b}}, \boldsymbol{b}\big)$, so that

$$
\begin{aligned}
&\bar{\boldsymbol{b}} - \boldsymbol{e}_1,\, \tilde{\boldsymbol{b}} - \boldsymbol{e}_2 \in C([0,\infty); \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1}(\mathbb{R}^3)),\\
&\boldsymbol{b} - \boldsymbol{e}_3 \in C([0,\infty); \dot{H}^{s_1+1} \cap \dot{H}^{s_2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1}(\mathbb{R}^3)),\\
&\boldsymbol{u} \in C([0,\infty); \dot{H}^{s_1+1} \cap \dot{H}^{s_2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2+1}(\mathbb{R}^3))\\
&\qquad \cap L^1(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)),\\
&\nabla p \in L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^3)).
\end{aligned}
\tag{2.8}
$$

Furthermore, there holds

$$
\begin{aligned}
&\|(\bar{\boldsymbol{b}} - \boldsymbol{e}_1, \tilde{\boldsymbol{b}} - \boldsymbol{e}_2)\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1}\cap \dot{H}^{s_2+1})} + \|\boldsymbol{b} - \boldsymbol{e}_3\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1}\cap \dot{H}^{s_2})}\\
&\quad + \|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1}\cap \dot{H}^{s_2})} + \|\boldsymbol{b} - \boldsymbol{e}_3\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+1}\cap \dot{H}^{s_2+1})}\\
&\quad + \|\boldsymbol{u}\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+2}\cap \dot{H}^{s_2+1})} + \|\boldsymbol{u}\|_{L^1(\mathbb{R}^+; \dot{H}^{s_1+2}\cap \dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla p\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1}\cap \dot{H}^{s_2})}\\
&\leq C\big(\|\nabla \boldsymbol{\Psi}\|_{\dot{B}^{s_2+\frac{3}{p}-\frac{3}{2}}_{p,2} \cap B^{s_1+\frac{3}{p}-\frac{1}{2}}_{p,1}} + \|\boldsymbol{u}_0\|_{\dot{H}^{s_2}\cap \dot{B}^{\frac{3}{p}-1}_{p,1}} + \|\nabla \boldsymbol{u}_0\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}}\big).
\end{aligned}
\tag{2.9}
$$

In order to avoid the difficulty of propagating anisotropic regularity for the transport equation in the system (2.6), we shall reformulate (2.6) in the Lagrangian coordinates. Toward this, we need first to find a volume preserving diffeomorphism $X_0(y)$ on $\mathbb{R}^3$ so that there holds (1.4).

Lemma 2.1. *Let $p \in (1,2)$, $s > 1 + \frac{3}{p}$. Let $\boldsymbol{\Psi} = (\psi_1, \psi_2, \psi_3)^T$ satisfy $\nabla \boldsymbol{\Psi} \in B^{s-1}_{p,1}(\mathbb{R}^3)$, $det\big(I - \nabla \boldsymbol{\Psi}\big) = 1$ and $\|\nabla \boldsymbol{\Psi}\|_{B^{s-1}_{p,1}} \leq \varepsilon_0$ for some ε_0 sufficiently small. Then for $U_0 = \big(I - \nabla \boldsymbol{\Psi}\big)^{-1}$, there exists $Y_0(y) = (Y_0^1(y), Y_0^2(y), Y_0^3(y))^T$ so that $X_0(y) = y + Y_0(y)$ satisfies*

$$
U_0 \circ X_0(y) = \nabla_y X_0(y) = I + \nabla_y Y_0(y) \quad \text{and} \quad \|\nabla_y Y_0\|_{B^{s-1}_{p,1}} \leq C \|\nabla_x \boldsymbol{\Psi}\|_{B^{s-1}_{p,1}}.
\tag{2.10}
$$

Proof. Let $Y = (Y^1, Y^2, Y^3)^T$, we denote

$$
\boldsymbol{F}(y, Y) \stackrel{\text{def}}{=} Y - \boldsymbol{\Psi}(y + Y),
\tag{2.11}
$$

with $\boldsymbol{F}(y,Y) = (F^1(y,Y), F^2(y,Y), F^3(y,Y))^T$.

It is easy to observe from the assumption: $\det(I - \nabla_x \boldsymbol{\Psi}) = \det U_0 = 1$, that

$$
\det \frac{\partial(F^1, F^2, F^3)}{\partial(Y^1, Y^2, Y^3)} = \det(I - \nabla_x \boldsymbol{\Psi})|_{x = y+Y} = 1,
$$

from which, $\|\nabla \boldsymbol{\Psi}\|_{L^\infty} \leq C\varepsilon_0$ for some ε_0 sufficiently small, and the classical implicit function theorem, we deduce that around every point y, the function $\boldsymbol{F}(y,Y) = \boldsymbol{0}$ determines a unique function $Y_0(y) = (Y_0^1(y), Y_0^2(y), Y_0^3(y))^T$ so that

$$
\boldsymbol{F}(y, Y_0(y)) = \boldsymbol{0},
$$

or equivalently

$$
Y_0(y) = \boldsymbol{\Psi}(y + Y_0(y)).
\tag{2.12}
$$

Then denoting by $X_0(y) = y + Y_0(y)$, we have

$$
\begin{aligned}
\partial_{y_1} Y_0^j(y) =& \partial_{x_1}\psi_j \circ X_0(y)(1 + \partial_{y_1} Y_0^1(y)) + \partial_{x_2}\psi_j \circ X_0(y)\partial_{y_1} Y_0^2(y)\\
&+ \partial_{x_3}\psi_j \circ X_0(y)\partial_{y_1} Y_0^3(y), \quad \text{for} \quad j = 1,2,3.
\end{aligned}
\tag{2.13}
$$

Due to the fact that $\det(I-\nabla_x\boldsymbol{\Psi})=\det U_0=1$, we conclude that $I-\nabla_x\boldsymbol{\Psi}$ equals the adjoint matrix of $U_0\stackrel{\text{def}}{=}\big(b_{ij}\big)_{i,j=1,2,3}$, which along with (2.13) ensures that

$$\partial_{y_1}Y_0^1(y)=b_{11}\circ X_0(y)-1,\quad \partial_{y_1}Y_0^2(y)=b_{21}\circ X_0(y),\quad \partial_{y_1}Y_0^3(y)=b_{31}\circ X_0(y). \tag{2.14}$$

Along the same line, one has

$$\partial_{y_j}Y_0^i(y)=b_{ij}\circ X_0(y)-\delta_{ij}.$$

which implies the first part of (2.10). This in particular leads to

$$\nabla_x\big(X_0^{-1}(x)\big)=\big((\nabla_yX_0)\circ X_0^{-1}(x)\big)^{-1}=U_0^{-1}(x)=I-\nabla\boldsymbol{\Psi}(x),$$

from which, (2.12), $\|\nabla\boldsymbol{\Psi}\|_{B^{s-1}_{p,1}}\leq\varepsilon_0$ for $s>1+\frac{3}{p}$ and Lemma A.1, we achieve the second part of (2.10). □

With $X_0(y)=y+Y_0(y)$ obtained in Lemma 2.1, we now define the flow map $X(t,y)$ by

$$\begin{cases}\dfrac{dX(t,y)}{dt}=\boldsymbol{u}(t,X(t,y)),\\ X(t,y)|_{t=0}=X_0(y),\end{cases}$$

and $Y(t,y)$ through

$$X(t,y)=X_0(y)+\int_0^t\boldsymbol{u}(s,X(s,y))\,ds\stackrel{\text{def}}{=}y+Y(t,y). \tag{2.15}$$

Then by virtue of Proposition 1.8 of [23] and (2.10), we deduce from (2.6) that

$$U(t,X(t,y))=\nabla_yX(t,y)=I+\nabla_yY(t,y)\quad\text{and}\quad\det\big(I+\nabla_yY(t,y)\big)=1. \tag{2.16}$$

Denoting $U(t,X(t,y))\stackrel{\text{def}}{=}(a_{ij})_{i,j=1,2,3}$ and $\mathcal{A}_Y\stackrel{\text{def}}{=}(b_{ij})_{i,j=1,2,3}$ with

$$\begin{aligned}
&b_{11}=(1+\partial_2Y^2)(1+\partial_3Y^3)-\partial_3Y^2\partial_2Y^3, && b_{12}=\partial_3Y^1\partial_2Y^3-\partial_2Y^1(1+\partial_3Y^3),\\
&b_{13}=\partial_2Y^1\partial_3Y^2-\partial_3Y^1(1+\partial_2Y^2), && b_{21}=\partial_3Y^2\partial_1Y^3-\partial_1Y^2(1+\partial_3Y^3),\\
&b_{22}=(1+\partial_1Y^1)(1+\partial_3Y^3)-\partial_3Y^1\partial_1Y^3, && b_{23}=\partial_3Y^1\partial_1Y^2-(1+\partial_1Y^1)\partial_3Y^2,\\
&b_{31}=\partial_1Y^2\partial_2Y^3-(1+\partial_2Y^2)\partial_1Y^3, && b_{32}=\partial_2Y^1\partial_1Y^3-(1+\partial_1Y^1)\partial_2Y^3,\\
&b_{33}=(1+\partial_1Y^1)(1+\partial_2Y^2)-\partial_2Y^1\partial_1Y^2.
\end{aligned} \tag{2.17}$$

It is easy to observe that $\sum_{i=1}^3\frac{\partial b_{ij}}{\partial y_i}=0$ (see also Lemma 2.1 of [27]). Moreover, as $\det U=1$, $\mathcal{A}_Y=(I+\nabla_yY)^{-1}$. Then it follows from (2.16) that

$$\boldsymbol{b}\circ X(t,y)=(\partial_{y_3}Y^1,\partial_{y_3}Y^2,1+\partial_{y_3}Y^3)^T\quad\text{and}\quad\mathcal{A}_Y(\boldsymbol{b}\circ X)=(0,0,1)^T, \tag{2.18}$$

from which, we infer

$$\begin{aligned}(\boldsymbol{b}\cdot\nabla_x\boldsymbol{b})\circ X(t,y)&=[\mathrm{div}_x(\boldsymbol{b}\otimes\boldsymbol{b})]\circ X(t,y)\\ &=\nabla_y\cdot[\mathcal{A}_Y(\boldsymbol{b}\circ X)\otimes(\boldsymbol{b}\circ X)]=\partial_{y_3}(\boldsymbol{b}\circ X)=\partial_{y_3}^2Y(t,y).\end{aligned} \tag{2.19}$$

Thanks to (2.15) and (2.19), we can equivalently reformulate (2.6) as

$$\begin{cases}Y_{tt}-\nabla_Y\cdot\nabla_YY_t-\partial_{y_3}^2Y+\nabla_Yq=\mathbf{0},\\ \nabla_Y\cdot Y_t=0,\\ Y|_{t=0}=Y_0,\quad Y_t|_{t=0}=\boldsymbol{u}_0\circ X_0(y)\stackrel{\text{def}}{=}Y_1,\end{cases} \tag{2.20}$$

where $q(t,y) \stackrel{\text{def}}{=} (p+\frac{1}{2}|\boldsymbol{b}|^2)\circ X(t,y)$ and $\nabla_Y \stackrel{\text{def}}{=} \mathcal{A}_Y^T\nabla_y$ with $\mathcal{A}_Y=(b_{ij})_{i,j=1,2,3}$ being determined by (2.17). Here and in what follows, we always assume that $\|\nabla_y Y\|_{L^\infty}\leq \frac{1}{2}$. Under this assumption, we rewrite (2.20) as

$$
\begin{cases} Y_{tt}-\Delta_y Y_t-\partial_{y_3}^2 Y=\boldsymbol{f}(Y,q),\\ \nabla_y\cdot Y=\rho(Y),\\ Y|_{t=0}=Y_0,\quad Y_t|_{t=0}=Y_1. \end{cases} \tag{2.21}
$$

where

$$
\begin{aligned}
\boldsymbol{f}(Y,q)=&(\nabla_Y\cdot\nabla_Y-\Delta_y)Y_t-\nabla_Y q,\\
\rho(Y)=&\nabla_y\cdot Y_0-\int_0^t(\nabla_Y-\nabla_y)\cdot Y_s\,ds\\
=&-\sum_{i<j}\partial_iY^i\partial_jY^j+\sum_{i<j}\partial_iY^j\partial_jY^i-\partial_1Y^1\partial_2Y^2\partial_3Y^3-\partial_3Y^1\partial_1Y^2\partial_2Y^3\\
&-\partial_2Y^1\partial_3Y^2\partial_1Y^3+\partial_1Y^1\partial_3Y^2\partial_2Y^3+\partial_3Y^1\partial_2Y^2\partial_1Y^3+\partial_2Y^1\partial_1Y^2\partial_3Y^3.
\end{aligned} \tag{2.22}
$$

Here we used (2.17) and $\det(I+\nabla Y_0)=1$ to derive the second equality of (2.22). Indeed thanks to (2.17), one has

$$
\begin{aligned}
(\nabla_Y-\nabla_y)\cdot Y_t=&\frac{d}{dt}\Big(\sum_{i<j}\partial_iY^i\partial_jY^j-\sum_{i<j}\partial_iY^j\partial_jY^i+\partial_1Y^1\partial_2Y^2\partial_3Y^3\\
&+\partial_3Y^1\partial_1Y^2\partial_2Y^3+\partial_2Y^1\partial_3Y^2\partial_1Y^3-\partial_1Y^1\partial_3Y^2\partial_2Y^3\\
&-\partial_3Y^1\partial_2Y^2\partial_1Y^3-\partial_2Y^1\partial_1Y^2\partial_3Y^3\Big)\\
=&\frac{d}{dt}\big(\det\big(I+\nabla_yY\big)-1-\nabla_y\cdot Y\big),
\end{aligned} \tag{2.23}
$$

which together with $\det(I+\nabla_yY_0)=1$ ensures the second equality of (2.22). Moreover, the equation $\nabla_y\cdot Y=\rho(Y)$ implies that $\det(I+\nabla_yY)=1$ and $\nabla_Y\cdot Y_t=0$.

For notational convenience, we shall neglect the subscripts x or y in ∂,∇ and Δ in the sequel. We make the convention that whenever ∇ acts on $(U,\boldsymbol{u},p)$, we understand $(\nabla U,\nabla\boldsymbol{u},\nabla p)$ as $(\nabla_xU,\nabla_x\boldsymbol{u},\nabla_xp)$. While ∇ acts on (Y,q), we understand $(\nabla Y,\nabla q)$ as (∇_yY,∇_yq). Similar conventions for ∂ and Δ.

For (2.21)-(2.22), we have the following global wellposedness result:

Theorem 2.2. *Let $s_1>\frac{5}{4}$, $s_2\in(-\frac{1}{2},-\frac{1}{4})$. Let (Y_0,Y_1) satisfy $(\partial_3Y_0,\,\Delta Y_0)\in\dot{H}^{s_1}\cap\dot{H}^{s_2}\cap\mathcal{B}^{\frac{1}{2},0}\cap\mathcal{B}^{s_1,0}(\mathbb{R}^3)$, $Y_1\in\dot{H}^{s_1+1}\cap\dot{H}^{s_2}\cap\mathcal{B}^{\frac{1}{2},0}\cap\mathcal{B}^{s_1,0}(\mathbb{R}^3)$ and*

$$
\det(I+\nabla Y_0)=1,\quad \nabla_{Y_0}\cdot Y_1=0,\quad \textit{and} \tag{2.24}
$$

$$
\begin{aligned}
&\|Y_0\|_{\dot{H}^{s_1+2}\cap\dot{H}^{s_2+2}}+\|\partial_3Y_0\|_{\dot{H}^{s_2}}+\|Y_1\|_{\dot{H}^{s_1+1}\cap\dot{H}^{s_2}}\\
&\quad+\|Y_0\|_{\mathcal{B}^{\frac{5}{2},0}\cap\mathcal{B}^{s_1+2,0}}+\|\partial_3Y_0\|_{\mathcal{B}^{\frac{1}{2},0}\cap\mathcal{B}^{s_1,0}}+\|Y_1\|_{\mathcal{B}^{\frac{1}{2},0}\cap\mathcal{B}^{s_1,0}}\leq\varepsilon_0
\end{aligned} \tag{2.25}
$$

for some ε_0 *sufficiently small. Then* (2.21)-(2.22) *has a unique global solution* (Y,q) *(up to a constant for* q*) so that*

$$
\begin{aligned}
&Y\in C([0,\infty);\dot H^{s_1+2}\cap\dot H^{s_2+2}\cap\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s_1+2,0}(\mathbb{R}^3))\quad\text{and}\\
&\partial_3Y\in C([0,\infty);\dot H^{s_2}\cap\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s_1,0}(\mathbb{R}^3))\cap L^2(\mathbb{R}^+;\dot H^{s_1+1}\cap\dot H^{s_2+1}(\mathbb{R}^3)),\\
&Y_t\in C([0,\infty);\dot H^{s_1+1}\cap\dot H^{s_2}\cap\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s_1,0}(\mathbb{R}^3))\cap L^2(\mathbb{R}^+;\dot H^{s_1+2}\cap\dot H^{s_2+1}(\mathbb{R}^3))\\
&\qquad\cap L^1(\mathbb{R}^+;\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s_1+2,0}(\mathbb{R}^3)),\\
&\nabla q\in L^2(\mathbb{R}^+;\dot H^{s_1}\cap\dot H^{s_2}(\mathbb{R}^3))\cap L^1(\mathbb{R}^+;\dot H^{s_1}\cap\dot H^{s_2}(\mathbb{R}^3)).
\end{aligned}\tag{2.26}
$$

Moreover, there hold $\det(I+\nabla Y)=1$, $\nabla_Y\cdot Y_t=0$, *and*

$$
\begin{aligned}
&\|Y\|^2_{L^\infty_T(\dot H^{s_1+2}\cap\dot H^{s_2+2})}+\|\partial_3Y\|^2_{L^\infty_T(\dot H^{s_2})}+\|Y_t\|^2_{L^\infty_T(\dot H^{s_1+1}\cap\dot H^{s_2})}+\|\partial_3Y\|^2_{L^2_T(\dot H^{s_1+1}\cap\dot H^{s_2+1})}\\
&\quad+\|Y_t\|^2_{L^2_T(\dot H^{s_1+2}\cap\dot H^{s_2+1})}+\|\nabla q\|^2_{L^2_T(\dot H^{s_1}\cap\dot H^{s_2})}+\|\nabla q\|^2_{L^1_T(\dot H^{s_1}\cap\dot H^{s_2})}\\
&\quad+\|Y\|^2_{L^\infty_T(\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s_1+2,0})}+\|\partial_3Y\|^2_{L^\infty_T(\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s_1,0})}+\|Y_t\|^2_{L^1_T(\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s_1+2,0})}\\
&\leq C\big(\|\partial_3Y_0\|^2_{\dot H^{s_2}}+\|Y_0\|^2_{\dot H^{s_1+2}\cap\dot H^{s_2+2}}+\|Y_1\|^2_{\dot H^{s_1+1}\cap\dot H^{s_2}}\\
&\quad+\|\partial_3Y_0\|^2_{\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s_1,0}}+\|Y_0\|^2_{\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s_1+2,0}}+\|Y_1\|^2_{\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s_1,0}}\big).
\end{aligned}\tag{2.27}
$$

Remark 2.1. *The norm of* $\|\cdot\|_{\mathcal{B}^{s,0}}$ *is given by Definition 3.2. We should mention once again that the equation* $\nabla\cdot Y=\rho(Y)$ *in* (2.21) *plays a key role in the proof of Theorem 2.2. In particular, we need to use this equation to derive the globally* L^1 *in time estimates of* ∇q *and* ∇Y_t*, which will be crucial for us to close the energy estimates for* (2.21)-(2.22).

Scheme of the proof and organization of the paper.

To avoid the difficulty caused by propagating anisotropic regularity for the transport equation in (2.6), we shall first prove the global wellposedness of the Lagrangian formulation (2.21)-(2.22) with small initial data.

Let (Y,q) be a smooth enough solution of (2.21), applying standard energy estimate to (2.21) leads to

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\frac12\big(\|Y_t\|^2_{\dot H^s}+\|Y_t\|^2_{\dot H^{s+1}}+\|\partial_3Y\|^2_{\dot H^s}+\|\partial_3Y\|^2_{\dot H^{s+1}}+\frac14\|Y\|^2_{\dot H^{s+2}}\big)\\
&\qquad-\frac14(Y_t\mid\Delta Y)_{\dot H^s}\Big\}+\frac34\|Y_t\|^2_{\dot H^{s+1}}+\|Y_t\|^2_{\dot H^{s+2}}+\frac14\|\partial_3Y\|^2_{\dot H^{s+1}}\\
&\quad=\big(\boldsymbol{f}\mid Y_t-\frac14\Delta Y-\Delta Y_t\big)_{\dot H^s}.
\end{aligned}\tag{2.28}
$$

where $(a\mid b)_{\dot H^s}$ denotes the standard $\dot H^s$ inner product of a and b. (2.28) shows that ∂_3Y belongs to $L^2(\mathbb{R}^+;\dot H^{s+1}(\mathbb{R}^3))$, however, there is no time dissipative estimate of ΔY. Therefore, in order to close the energy estimate in (2.28), we would require the source term $\boldsymbol{f}$ in (2.21) belonging to $L^1(\mathbb{R}^+;\dot H^s(\mathbb{R}^3))$. To achieve this, we need also the $L^1(\mathbb{R}^+;\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s_1+2,0}(\mathbb{R}^3))$ estimate of Y_t. Toward this, we shall use the dissipative estimates for ∂_3Y as well as the fact that $\nabla\cdot Y=\rho(Y)$ in a rather crucial way.

In the first part of Section 3, we shall present a heuristic analysis to the linearized system of (2.21)-(2.22), which motivates us to use anisotropic Littlewood-Paley theory below, then we shall collect some basic facts on functional framework and Littlewood-Paley analysis in Subsection 3.2.

In Section 4, we apply anisotropic Littlewood-Paley theory to explore the dissipative mechanism for a linearized model of (2.21)-(2.22).

In Section 5, we present the proof of Theorem 2.2, and we present the proof of Theorems 2.1 and 1.1 in Section 6.

Finally, we present the proofs of some technical lemmas in the Appendices.

3. Preliminary

3.1. **Spectral analysis to the linearized system of** (2.21)-(2.22)**.** We first investigate heuristically the spectrum properties to the following linearized system of (2.21)-(2.22):

$$\begin{cases} Y_{tt} - \Delta Y_t - \partial_3^2 Y = \boldsymbol{f}, \\ Y|_{t=0} = Y_0, \quad Y_t|_{t=0} = Y_1. \end{cases} \tag{3.1}$$

Note that the symbolic equation corresponds to (3.1) reads

$$\lambda^2 + |\xi|^2\lambda + \xi_3^2 = 0 \quad \text{for} \quad \xi = (\xi_h, \xi_3) \quad \text{and} \quad \xi_h = (\xi_1, \xi_2).$$

It is easy to calculate that this equation has two different eigenvalues

$$\lambda_\pm = -\frac{|\xi|^2 \pm \sqrt{|\xi|^4 - 4\xi_3^2}}{2}. \tag{3.2}$$

The Fourier modes correspond to λ_+ decays like $e^{-t|\xi|^2}$. Whereas the decay property of the Fourier modes corresponding to λ_- varies with directions of ξ as

$$\lambda_-(\xi) = -\frac{2\xi_3^2}{|\xi|^2\big(1+\sqrt{1-\frac{4\xi_3^2}{|\xi|^4}}\big)} \to -1 \quad \text{as} \quad |\xi| \to \infty \tag{3.3}$$

only in the ξ_3 direction. This shows that smooth solution of (3.1) decays in a very subtle way. In order to capture this delicate decay property for the linear equation (3.1), we shall decompose our frequency space into two parts: $\big\{\xi = (\xi_h, \xi_3) : \ |\xi|^2 \leq 2|\xi_3| \ \big\}$ and $\big\{\xi = (\xi_h, \xi_3) : \ |\xi|^2 > 2|\xi_3| \ \big\}$.

This heuristic analysis shows that the dissipative properties of the solutions to (3.1) may be more complicated than that for the linearized system of isentropic compressible Navier-Stokes system in [12], and this brief analysis also suggests us to employ the tool of anisotropic Littlewood-Paley theory as in [22] for 2-D incompressible MHD system and [21] for a modified 3-D MHD system, which has also been used in the study of the global wellposedness to 3-D anisotropic incompressible Navier-Stokes equations [7, 8, 9, 15, 16, 24, 25, 28]. One may check Section 4 below for the detailed rigorous analysis corresponding to this scenario.

3.2. **Littlewood-Paley theory.** The proof of Theorem 2.2 requires a dyadic decomposition of the Fourier variables, or the Littlewood-Paley decomposition. Let us briefly explain how it may be built in the case $x \in \mathbb{R}^3$ (see e.g. [2]). Let φ and χ be smooth functions supported in $\mathcal{C} \stackrel{\text{def}}{=} \{\tau \in \mathbb{R}^+, \ \frac{3}{4} \leq \tau \leq \frac{8}{3}\}$ and $\mathcal{B} \stackrel{\text{def}}{=} \{\tau \in \mathbb{R}^+, \ \tau \leq \frac{4}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\tau) = 1 \quad \text{for} \quad \tau > 0 \quad \text{and} \quad \chi(\tau) + \sum_{j\geq 0} \varphi(2^{-j}\tau) = 1 \quad \text{for} \quad \tau \geq 0.$$

For $a \in \mathcal{S}'(\mathbb{R}^2)$, we set

$$
\begin{aligned}
&\Delta_k^h a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\widehat{a}), \qquad && S_k^h a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-k}|\xi_h|)\widehat{a}),\\
&\Delta_\ell^v a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}), && S_\ell^v a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\widehat{a}), \quad \text{and}\\
&\Delta_j a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}), && S_j a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}),
\end{aligned}
\tag{3.4}
$$

where $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of the distribution a. The dyadic operators satisfy the property of almost orthogonality:

$$
\Delta_k\Delta_j a \equiv 0 \quad \text{if} \quad |k-j| \geq 2 \quad \text{and} \quad \Delta_k(S_{j-1}a\Delta_j b) \equiv 0 \quad \text{if} \quad |k-j| \geq 5. \tag{3.5}
$$

Similar properties hold for Δ_k^h and Δ_ℓ^v.

Definition 3.1. *[Definition 2.15 of [2]] Let $(p,r) \in [1,+\infty]^2$, $s \in \mathbb{R}$ and $u \in \mathcal{S}_h'(\mathbb{R}^3)$, (see Definition 1.26 of [2]), which means $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\lim_{j\to-\infty} \|\chi(2^{-j}D)u\|_{L^\infty} = 0$, we set*

$$
\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \Big(2^{js}\|\Delta_j u\|_{L^p}\Big)_{\ell^r}.
$$

- *For $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define $\dot{B}^s_{p,r}(\mathbb{R}^3) \stackrel{\text{def}}{=} \big\{u \in \mathcal{S}_h'(\mathbb{R}^3) \bigm| \|u\|_{\dot{B}^s_{p,r}} < \infty\big\}$.*
- *If $k \in \mathbb{N}$ and $\frac{3}{p} + k - 1 \leq s < \frac{3}{p} + k$ (or $s = \frac{3}{p} + k$ if $r = 1$), then $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined as the subset of distributions $u \in \mathcal{S}_h'(\mathbb{R}^3)$ such that $\partial^\beta u \in \dot{B}^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta| = k$.*

Inhomogenous Besov spaces $B^s_{p,r}(\mathbb{R}^3)$ can be defined similarly (see Definition 2.68 of [2]). For simplicity, we shall abbreviate $\dot{B}^s_{2,1}(\mathbb{R}^3)$ (resp. $B^s_{2,1}(\mathbb{R}^3)$) as $\dot{B}^s(\mathbb{R}^3)$ (resp. $B^s(\mathbb{R}^3)$) in all that follows.

Remark 3.1. *(1) It is easy to observe that $\dot{B}^s_{2,2}(\mathbb{R}^3) = \dot{H}^s(\mathbb{R}^3)$.*

(2) Let $(p,r) \in [1,+\infty]^2$, $s \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^3)$. Then $u \in \dot{B}^s_{p,r}(\mathbb{R}^3)$ if and only if there exists $\{c_{j,r}\}_{j\in\mathbb{Z}}$ such that $\|c_{j,r}\|_{\ell^r} = 1$ and

$$
\|\Delta_j u\|_{L^p} \leq C c_{j,r} 2^{-js}\|u\|_{\dot{B}^s_{p,r}} \quad \text{for all} \quad j \in \mathbb{Z}.
$$

(3) Let $s, s_1, s_2 \in \mathbb{R}$ with $s_1 < s < s_2$ and $u \in \dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^3)$. Then $u \in \dot{B}^s(\mathbb{R}^3)$, and there holds

$$
\|u\|_{\dot{B}^s} \lesssim \|u\|_{\dot{H}^{s_1}}^{\frac{s_2-s}{s_2-s_1}} \|u\|_{\dot{H}^{s_2}}^{\frac{s-s_1}{s_2-s_1}} \lesssim \|u\|_{\dot{H}^{s_1}} + \|u\|_{\dot{H}^{s_2}}. \tag{3.6}
$$

For the convenience of the readers, we recall the following Bernstein type lemma from [2, 9, 24]:

Lemma 3.1. *Let $\mathcal{B}_h$ (resp. $\mathcal{B}_v$) be a ball of $\mathbb{R}^2$ (resp. $\mathbb{R}$), and $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) a ring of $\mathbb{R}^2$ (resp. $\mathbb{R}$); let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then there holds:*
If the support of $\widehat{a}$ is included in $2^k\mathcal{B}_h$, then

$$
\|\partial_h^\alpha a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k\left(|\alpha|+2\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\right)}\|a\|_{L_h^{p_2}(L_v^{q_1})}, \quad \text{for} \quad \partial_h = (\partial_1, \partial_2).
$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{B}_v$, then

$$
\|\partial_3^\beta a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{\ell\left(\beta+\left(\frac{1}{q_2}-\frac{1}{q_1}\right)\right)}\|a\|_{L_h^{p_1}(L_v^{q_2})}.
$$

If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_h$, then

$$
\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN}\|\partial_h^N a\|_{L_h^{p_1}(L_v^{q_1})}.
$$

If the support of $\widehat{a}$ *is included in* $2^\ell\mathcal{C}_v$, *then*

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-\ell N}\|\partial_3^N a\|_{L_h^{p_1}(L_v^{q_1})}.$$

In order to obtain the $L^1(\mathbb{R}^+;\mathrm{Lip}(\mathbb{R}^3))$ estimate of Y_t for the linearized equation (3.1), we recall the following anisotropic Besov type space from [21, 22]:

Definition 3.2. *Let* $s_1, s_2 \in \mathbb{R}$ *and* $u \in \mathcal{S}_h'(\mathbb{R}^3)$, *we define the norm*

$$\|u\|_{\mathcal{B}^{s_1,s_2}} \stackrel{\text{def}}{=} \sum_{j,k\in\mathbb{Z}^2} 2^{js_1}2^{ks_2}\|\Delta_j\Delta_k^v u\|_{L^2}.$$

Then we have the following three dimensional version of Lemma 3.2 in [22]:

Lemma 3.2. *Let* $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$, *which satisfy* $s_1 < \tau_1 + \tau_2 < s_2$ *and* $\tau_2 > 0$. *Let* $a \in \dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^3)$. *Then* $a \in \mathcal{B}^{\tau_1,\tau_2}(\mathbb{R}^3)$, *and there holds*

$$\|a\|_{\mathcal{B}^{\tau_1,\tau_2}} \lesssim \|a\|_{\dot{B}^{\tau_1+\tau_2}} \lesssim \|a\|_{\dot{H}^{s_1}} + \|a\|_{\dot{H}^{s_2}}.$$

Proof. By virtue of Definition 3.2 and the fact: $j \geq k - N_0$ for some fixed positive integer N_0 in dyadic operator $\Delta_j\Delta_k^v$, we infer

$$\begin{aligned}\|a\|_{\mathcal{B}^{\tau_1,\tau_2}} = \sum_{\substack{j,k\in\mathbb{Z}^2\\ k\leq j+N_0}} 2^{j\tau_1}2^{k\tau_2}\|\Delta_j\Delta_k^v a\|_{L^2} &\lesssim \sum_{j\in\mathbb{Z}} 2^{j\tau_1}\|\Delta_j a\|_{L^2}\sum_{k\leq j+N_0}2^{k\tau_2}\\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(\tau_1+\tau_2)}\|\Delta_j a\|_{L^2} \lesssim \|a\|_{\dot{B}^{\tau_1+\tau_2}},\end{aligned}$$

which together with (3.6) completes the proof of the lemma. ☐

In order to obtain a better description of the regularizing effect for the transport-diffusion equation, we will use Chemin-Lerner type spaces $\widetilde{L}^q_T(B^s_{p,r}(\mathbb{R}^3))$ (see [2] for instance).

Definition 3.3. *Let* $(r, q, p) \in [1, +\infty]^3$ *and* $T \in (0, +\infty]$. *We define the* $\widetilde{L}^q_T(\dot{B}^s_{p,r}(\mathbb{R}^3))$ *and* $\widetilde{L}^q_T(\mathcal{B}^{s_1,s_2}(\mathbb{R}^3))$ *by*

$$\|u\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} \stackrel{def}{=} \Big(\sum_{j\in\mathbb{Z}} 2^{jrs}\|\Delta_j u\|^r_{L^q_T(L^p)}\Big)^{\frac{1}{r}}, \quad \|u\|_{\widetilde{L}^q_T(\mathcal{B}^{s_1,s_2})} \stackrel{def}{=} \sum_{j,k\in\mathbb{Z}^2} 2^{js_1}2^{ks_2}\|\Delta_j\Delta_k^v u\|_{L^q_T(L^2)},$$

with the usual change if $r = \infty$.

Remark 3.2. *The proof of Lemma 3.2 ensures that*

$$\|u\|_{\widetilde{L}^2_T(\mathcal{B}^{\tau_1,\tau_2})} \lesssim \|u\|_{\widetilde{L}^2_T(\dot{B}^{\tau_1+\tau_2})} \lesssim \|u\|_{L^2_T(\dot{H}^{s_1})} + \|u\|_{L^2(\dot{H}^{s_2})}, \tag{3.7}$$

for τ_1, τ_2 *and* s_1, s_2 *given by Lemma 3.2.*

We also recall the isotropic para-differential decomposition of Bony from [3]: let $a, b \in \mathcal{S}'(\mathbb{R}^3)$,

$$\begin{aligned}&ab = T(a,b) + \mathcal{R}(a,b), \quad \text{or} \quad ab = T(a,b) + \bar{T}(a,b) + R(a,b), \quad \text{where}\\ &T(a,b) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} S_{j-1}a\Delta_j b, \quad \bar{T}(a,b) \stackrel{\text{def}}{=} T(b,a), \quad \mathcal{R}(a,b) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \Delta_j a S_{j+2}b, \quad \text{and}\\ &R(a,b) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \Delta_j a\tilde{\Delta}_j b, \quad \text{with} \quad \tilde{\Delta}_j b \stackrel{\text{def}}{=} \sum_{\ell=j-1}^{j+1}\Delta_\ell b.\end{aligned} \tag{3.8}$$

Considering the special structure of the functions in $\mathcal{B}^{s_1,s_2}(\mathbb{R}^3)$, we sometime use both isentropic Bony's decomposition(3.8) and (3.8) for the vertical variable x_3 simultaneously.

As an application of the above basic facts on Littlewood-Paley theory, we present the following product laws in space $\mathcal{B}^{s_1,s_2}(\mathbb{R}^3)$.

Lemma 3.3. *Let $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$, which satisfy $s_1, s_2 \leq 1$, $\tau_1, \tau_2 \leq \frac{1}{2}$ and $s_1 + s_2 > 0$, $\tau_1 + \tau_2 > 0$. Then for $a \in \mathcal{B}^{s_1,\tau_1}(\mathbb{R}^3)$ and $b \in \mathcal{B}^{s_2,\tau_2}(\mathbb{R}^3)$, $ab \in \mathcal{B}^{s_1+s_2-1,\tau_1+\tau_2-\frac{1}{2}}(\mathbb{R}^3)$ and there holds*

$$\|ab\|_{\mathcal{B}^{s_1+s_2-1,\tau_1+\tau_2-\frac{1}{2}}} \lesssim \|a\|_{\mathcal{B}^{s_1,\tau_1}} \|b\|_{\mathcal{B}^{s_2,\tau_2}}. \tag{3.9}$$

Proof. The proof of this lemma is identical to that of Lemma 3.3 in [22], we omit the details here. □

Lemma 3.4. *Let $\delta \in [0, \frac{1}{2})$, $s_1 \leq \frac{3}{2} - \delta$, $s_2 \leq 1 + \delta$ and $s_1 + s_2 > \frac{1}{2}$. Then one has*

$$\|ab\|_{\mathcal{B}^{s_1+s_2-\frac{3}{2},0}} \lesssim \|a\|_{\dot{B}^{s_1}} \|b\|_{\mathcal{B}^{s_2-\delta,\delta}}.$$

Proof. By virtue of Lemma 3.2 and Lemma 3.3, we have

$$\|ab\|_{\mathcal{B}^{s_1+s_2-\frac{3}{2},0}} \lesssim \|a\|_{\mathcal{B}^{s_1-\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \|b\|_{\mathcal{B}^{s_2-\delta,\delta}} \lesssim \|a\|_{\dot{B}^{s_1}} \|b\|_{\mathcal{B}^{s_2-\delta,\delta}}.$$

This completes the proof of the lemma. □

Remark 3.3. *It follows from Lemma 3.3 and Lemma 3.4 that*

$$\begin{aligned} &\|ab\|_{\mathcal{B}^{s,0}} \lesssim \|a\|_{\dot{B}^{\frac{3}{2}}} \|b\|_{\mathcal{B}^{s,0}}, \quad \text{and} \quad \|ab\|_{\mathcal{B}^{s,0}} \lesssim \|a\|_{\dot{B}^{\frac{5}{4}}} \|b\|_{\dot{B}^{s+\frac{1}{4}}} \quad \text{for} \quad -1 < s \leq 1, \\ &\|ab\|_{\mathcal{B}^{s,0}} \lesssim \|a\|_{\dot{B}^{\frac{1}{2}}} \|b\|_{\dot{B}^{s+1}} \quad \text{for} \quad -1 < s \leq 1. \end{aligned} \tag{3.10}$$

Lemma 3.5. *For any $s > -1$, there holds*

$$\begin{aligned} &\|ab\|_{\mathcal{B}^{s,0}} \lesssim \|a\|_{\dot{B}^{\frac{3}{2}}} \|b\|_{\mathcal{B}^{s,0}} + \|b\|_{\dot{B}^{\frac{3}{2}}} \|a\|_{\mathcal{B}^{s,0}}, \\ &\|ab\|_{\mathcal{B}^{s,0}} \lesssim \|a\|_{\dot{B}^{\frac{3}{2}}} \|b\|_{\mathcal{B}^{s,0}} + \|b\|_{\mathcal{B}^{1,0}} \|a\|_{\mathcal{B}^{s,\frac{1}{2}}}, \end{aligned} \tag{3.11}$$

and

$$\|ab\|_{\mathcal{B}^{s,0}} \lesssim \|a\|_{\dot{B}^{\frac{3}{2}-\delta_1}} \|b\|_{\dot{B}^{s+\delta_1}} + \|b\|_{\dot{B}^{\frac{3}{2}-\delta_2}} \|a\|_{\dot{B}^{s+\delta_2}}, \tag{3.12}$$

for $\delta_1, \delta_2 \in (0, \frac{1}{2})$.

Proof. We first get, by using Bony's decomposition (3.8) and (3.8) for the vertical variable, that

$$ab = \big(TT^v + T\bar{T}^v + TR^v + \bar{T}T^v + \bar{T}\bar{T}^v + \bar{T}R^v + RT^v + R\bar{T}^v + RR^v\big)(a,b). \tag{3.13}$$

We shall present the detailed estimates to typical terms above. Indeed applying Lemma 3.1 gives

$$\begin{aligned} \|\Delta_j \Delta_k^v (TR^v(a,b))\|_{L^2} &\lesssim 2^{\frac{k}{2}} \sum_{\substack{|j'-j|\leq 4 \\ k' \geq k-N_0}} \|S_{j'-1}\Delta_{k'}^v a\|_{L_h^\infty(L_v^2)} \|\Delta_{j'} \widetilde{\Delta}_{k'}^v b\|_{L^2} \\ &\lesssim 2^{\frac{k}{2}} \sum_{\substack{|j'-j|\leq 4 \\ k' \geq k-N_0}} d_{j',k'} 2^{-j's} 2^{-\frac{k'}{2}} \|a\|_{\mathcal{B}^{1,\frac{1}{2}}} \|b\|_{\mathcal{B}^{s,0}} \lesssim d_{j,k} 2^{-js} \|a\|_{\dot{B}^{\frac{3}{2}}} \|b\|_{\mathcal{B}^{s,0}}, \end{aligned}$$

as $\|S_{j'-1}\Delta_{k'}^v a\|_{L_h^\infty(L_v^2)} \lesssim 2^{-\frac{k'}{2}} \|a\|_{\mathcal{B}^{1,\frac{1}{2}}}$. Similar estimate holds for $\Delta_j \Delta_k^v (\bar{T}R^v(a,b))$.

Along the same line, we have

$$\begin{aligned}\|\Delta_j\Delta_k^v(RR^v(a,b))\|_{L^2} &\lesssim 2^j2^{\frac{k}{2}}\sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}}\|\Delta_{j'}\Delta_{k'}^v a\|_{L^2}\|\widetilde{\Delta}_{j'}\widetilde{\Delta}_{k'}^v b\|_{L^2}\\ &\lesssim 2^j2^{\frac{k}{2}}\sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}}d_{j',k'}2^{-j'(s+1)}2^{-\frac{k'}{2}}\|a\|_{\mathcal{B}^{1,\frac12}}\|b\|_{\mathcal{B}^{s,0}}\\ &\lesssim d_{j,k}2^{-js}\|a\|_{\dot{B}^{\frac32}}\|b\|_{\mathcal{B}^{s,0}}\end{aligned}$$

due to the fact: $s+1>0$. The estimate to the remaining terms in (3.13) is identical, and we omit the details here.

Whence thanks to (3.13), we arrive at

$$\|\Delta_j\Delta_k^v(ab)\|_{L^2}\lesssim d_{j,k}2^{-js}\big(\|a\|_{\dot{B}^{\frac32}}\|b\|_{\mathcal{B}^{s,0}}+\|b\|_{\dot{B}^{\frac32}}\|a\|_{\mathcal{B}^{s,0}}\big),$$

which implies the first inequality of (3.11). Exactly along the same line, we can prove the second inequality of (3.11). Finally notice from Lemma 3.2 that $\dot{B}^{\frac32-\delta}(\mathbb{R}^3)\hookrightarrow\mathcal{B}^{1,\frac12-\delta}(\mathbb{R}^3)$ and $\dot{B}^{s+\delta}(\mathbb{R}^3)\hookrightarrow\mathcal{B}^{s,\delta}(\mathbb{R}^3)$ for $\delta\in(0,\frac12)$, the proof of (3.12) is identical to that of (3.11), we omit the details here. This concludes the proof of Lemma 3.5. ☐

4. $L^1_T(\mathcal{B}^{s+2,0})$ ESTIMATE OF Y_t FOR $s=\frac12$ AND $s>1$

4.1. The estimate of $\|Y_t\|_{L^1_T(\mathcal{B}^{s+2,0})}$ for the linearized system (3.1).

Proposition 4.1. *Let Y be a smooth enough solution of (3.1) on $[0,T]$. Then for any $s\in\mathbb{R}$, there holds*

$$\begin{aligned}\|Y_t\|_{\widetilde{L}^\infty_T(\mathcal{B}^{s,0})}+\|\partial_3Y\|_{\widetilde{L}^\infty_T(\mathcal{B}^{s,0})}+\|Y\|_{\widetilde{L}^\infty_T(\mathcal{B}^{s+2,0})}+\|Y_t\|_{L^1_T(\mathcal{B}^{s+2,0})}&\\ +\|\partial_3Y\|_{\widetilde{L}^2_T(\mathcal{B}^{s+1,0})}\lesssim\|Y_1\|_{\mathcal{B}^{s,0}}+\|\partial_3Y_0\|_{\mathcal{B}^{s,0}}+\|Y_0\|_{\mathcal{B}^{s+2,0}}+\|\boldsymbol{f}\|_{L^1_T(\mathcal{B}^{s,0})}&.\end{aligned}\tag{4.1}$$

Proof. We first get, by applying $\Delta_j\Delta_k^v$ to (3.1), that

$$\Delta_j\Delta_k^vY_{tt}-\Delta\Delta_j\Delta_k^vY_t-\partial_3^2\Delta_j\Delta_k^vY=\Delta_j\Delta_k^v\boldsymbol{f}.\tag{4.2}$$

Taking the L^2 inner product of (4.2) with $\Delta_j\Delta_k^vY_t$ gives

$$\frac12\frac{d}{dt}\Big(\|\Delta_j\Delta_k^vY_t\|_{L^2}^2+\|\partial_3\Delta_j\Delta_k^vY\|_{L^2}^2\Big)+\|\nabla\Delta_j\Delta_k^vY_t\|_{L^2}^2=(\Delta_j\Delta_k^v\boldsymbol{f}\mid\Delta_j\Delta_k^vY_t).\tag{4.3}$$

While taking the L^2 inner product of (4.2) with $\Delta\Delta_j\Delta_k^vY$ leads to

$$(\Delta_j\Delta_k^vY_{tt}\mid\Delta\Delta_j\Delta_k^vY)-\frac12\frac{d}{dt}\|\Delta\Delta_j\Delta_k^vY\|_{L^2}^2-\|\partial_3\nabla\Delta_j\Delta_k^vY\|_{L^2}^2=(\Delta_j\Delta_k^v\boldsymbol{f}\mid\Delta\Delta_j\Delta_k^vY).$$

Notice that

$$(\Delta_j\Delta_k^vY_{tt}\mid\Delta\Delta_j\Delta_k^vY)=\frac{d}{dt}(\Delta_j\Delta_k^vY_t\mid\Delta\Delta_j\Delta_k^vY)-(\Delta_j\Delta_k^vY_t\mid\Delta\Delta_j\Delta_k^vY_t),$$

so that there holds

$$\begin{aligned}\frac{d}{dt}\Big(\frac12\|\Delta\Delta_j\Delta_k^vY\|_{L^2}^2-(\Delta_j\Delta_k^vY_t\mid\Delta\Delta_j\Delta_k^vY)\Big)&\\ -\|\nabla\Delta_j\Delta_k^vY_t\|_{L^2}^2+\|\partial_3\nabla\Delta_j\Delta_k^vY\|_{L^2}^2=-(\Delta_j\Delta_k^v\boldsymbol{f}\mid\Delta\Delta_j\Delta_k^vY)&.\end{aligned}\tag{4.4}$$

(4.3)+$\frac{1}{4}$(4.4) gives rise to

$$\frac{d}{dt}g_{j,k}^2(t)+\frac{3}{4}\|\nabla\Delta_j\Delta_k^vY_t\|_{L^2}^2+\frac{1}{4}\|\partial_3\nabla\Delta_j\Delta_k^vY\|_{L^2}^2 \\ =\big(\Delta_j\Delta_k^v\boldsymbol{f}\mid\Delta_j\Delta_k^vY_t-\frac{1}{4}\Delta\Delta_j\Delta_k^vY\big), \tag{4.5}$$

where

$$g_{j,k}^2(t)\stackrel{\text{def}}{=}\frac{1}{2}\big(\|\Delta_j\Delta_k^vY_t(t)\|_{L^2}^2+\|\partial_3\Delta_j\Delta_k^vY(t)\|_{L^2}^2 \\ +\frac{1}{4}\|\Delta\Delta_j\Delta_k^vY(t)\|_{L^2}^2\big)-\frac{1}{4}\big(\Delta_j\Delta_k^vY_t(t)\mid\Delta\Delta_j\Delta_k^vY(t)\big).$$

It is easy to observe that

$$g_{j,k}^2(t)\sim\|\Delta_j\Delta_k^vY_t(t)\|_{L^2}^2+\|\partial_3\Delta_j\Delta_k^vY(t)\|_{L^2}^2+\|\Delta\Delta_j\Delta_k^vY(t)\|_{L^2}^2. \tag{4.6}$$

With (4.5), (4.6), according to the heuristic discussions in Subsection 3.1 and similar to that in [21, 22], we shall separate the analysis of (4.5) into two cases: one is when $j\leq\frac{k+1}{2}$, and the other one is when $j>\frac{k+1}{2}$.

Case (1): $j\leq\frac{k+1}{2}$. In this case, we infer from Lemma 3.1 and (4.6) that

$$g_{j,k}^2(t)\sim\|\Delta_j\Delta_k^vY_t(t)\|_{L^2}^2+\|\partial_3\Delta_j\Delta_k^vY(t)\|_{L^2}^2,$$

and

$$\|\nabla\Delta_j\Delta_k^vY_t(t)\|_{L^2}^2+\|\partial_3\nabla\Delta_j\Delta_k^vY(t)\|_{L^2}^2 \\ \geq c2^{2j}\big(\|\Delta_j\Delta_k^vY_t(t)\|_{L^2}^2+\|\partial_3\Delta_j\Delta_k^vY(t)\|_{L^2}^2\big)\geq c2^{2j}g_{j,k}^2(t),$$

from which, for any $\varepsilon>0$, dividing (4.5) by $g_{j,k}(t)+\varepsilon$, then taking $\varepsilon\to0$ and integrating the resulting equation over $[0,T]$, we obtain

$$\|\Delta_j\Delta_k^vY_t\|_{L^\infty_T(L^2)}+\|\partial_3\Delta_j\Delta_k^vY\|_{L^\infty_T(L^2)}+\|\Delta\Delta_j\Delta_k^vY\|_{L^\infty_T(L^2)} \\ +c2^{2j}\big(\|\Delta_j\Delta_k^vY_t\|_{L^1_T(L^2)}+\|\partial_3\Delta_j\Delta_k^vY\|_{L^1_T(L^2)}\big) \\ \lesssim\|\Delta_j\Delta_k^vY_1\|_{L^2}+\|\partial_3\Delta_j\Delta_k^vY_0\|_{L^2}+\|\Delta_j\Delta_k^v\boldsymbol{f}\|_{L^1_T(L^2)}. \tag{4.7}$$

Case (2): $j>\frac{k+1}{2}$. Notice from Lemma 3.1 that in this case, one has

$$g_{j,k}^2(t)\sim\|\Delta_j\Delta_k^vY_t(t)\|_{L^2}^2+\|\Delta\Delta_j\Delta_k^vY(t)\|_{L^2}^2,$$

and

$$\|\nabla\Delta_j\Delta_k^vY_t(t)\|_{L^2}^2+\|\partial_3\nabla\Delta_j\Delta_k^vY(t)\|_{L^2}^2 \\ \geq c\frac{2^{2k}}{2^{2j}}\big(\|\Delta_j\Delta_k^vY_t(t)\|_{L^2}^2+\|\Delta\Delta_j\Delta_k^vY(t)\|_{L^2}^2\big)\geq c\frac{2^{2k}}{2^{2j}}g_{j,k}^2(t),$$

from which and (4.5), we deduce by a similar derivation of (4.7) that

$$\|\Delta_j\Delta_k^vY_t\|_{L^\infty_T(L^2)}+\|\partial_3\Delta_j\Delta_k^vY\|_{L^\infty_T(L^2)}+\|\Delta\Delta_j\Delta_k^vY\|_{L^\infty_T(L^2)} \\ +c\frac{2^{2k}}{2^{2j}}\big(\|\Delta_j\Delta_k^vY_t\|_{L^1_T(L^2)}+\|\Delta\Delta_j\Delta_k^vY\|_{L^1_T(L^2)}\big) \\ \lesssim\|\Delta_j\Delta_k^vY_1\|_{L^2}+\|\Delta\Delta_j\Delta_k^vY_0\|_{L^2}+\|\Delta_j\Delta_k^v\boldsymbol{f}\|_{L^1_T(L^2)}. \tag{4.8}$$

On the other hand, standard energy estimate applied to (4.2) yields that

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\Delta_k^vY_t(t)\|_{L^2}^2+\|\nabla\Delta_j\Delta_k^vY_t(t)\|_{L^2}^2=\big(\partial_3^2\Delta_j\Delta_k^vY+\Delta_j\Delta_k^v\boldsymbol{f}\mid\Delta_j\Delta_k^vY_t\big),$$

from which, Lemma 3.1 and (4.8), we infer

$$
\begin{aligned}
&\|\Delta_j\Delta_k^v Y_t\|_{L^\infty_T(L^2)}+c2^{2j}\|\Delta_j\Delta_k^v Y_t\|_{L^1_T(L^2)}\\
&\qquad\le \|\Delta_j\Delta_k^v Y_1\|_{L^2}+C\big(2^{2k}\|\Delta_j\Delta_k^v Y\|_{L^1_T(L^2)}+\|\Delta_j\Delta_k^v \boldsymbol{f}\|_{L^1_T(L^2)}\big)\\
&\qquad\lesssim \|\Delta_j\Delta_k^v Y_1\|_{L^2}+\|\Delta\Delta_j\Delta_k^v Y_0\|_{L^2}+\|\Delta_j\Delta_k^v \boldsymbol{f}\|_{L^1_T(L^2)}\quad\text{for}\quad j>\frac{k+1}{2}.
\end{aligned}\tag{4.9}
$$

Therefore according to Definitions 3.2 and 3.3, we get, by summing up (4.7), (4.8) and (4.9), that

$$
\begin{aligned}
&\|Y_t\|_{\widetilde{L}^\infty_T(\mathcal{B}^{s,0})}+\|\partial_3 Y\|_{\widetilde{L}^\infty_T(\mathcal{B}^{s,0})}+\|Y\|_{\widetilde{L}^\infty_T(\mathcal{B}^{s+2,0})}+\|Y_t\|_{L^1_T(\mathcal{B}^{s+2,0})}\\
&\qquad\qquad\lesssim \|Y_1\|_{\mathcal{B}^{s,0}}+\|\partial_3 Y_0\|_{\mathcal{B}^{s,0}}+\|Y_0\|_{\mathcal{B}^{s+2,0}}+\|\boldsymbol{f}\|_{L^1_T(\mathcal{B}^{s,0})}.
\end{aligned}\tag{4.10}
$$

On the other hand, it follows from (4.5) that

$$
\|\partial_3\nabla\Delta_j\Delta_k^v Y\|_{L^2_T(L^2)}\lesssim \|\Delta_j\Delta_k^v \boldsymbol{f}\|_{L^1_T(L^2)}+\|\Delta_j\Delta_k^v Y_t\|_{L^\infty_T(L^2)}+\|\Delta\Delta_j\Delta_k^v Y\|_{L^\infty_T(L^2)},
$$

so that

$$
\|\partial_3 Y\|_{\widetilde{L}^2_T(\mathcal{B}^{s+1,0})}\lesssim \|\boldsymbol{f}\|_{L^1_T(\mathcal{B}^{s,0})}+\|Y\|_{\widetilde{L}^\infty_T(\mathcal{B}^{s,0})}+\|Y\|_{\widetilde{L}^\infty_T(\mathcal{B}^{s+2,0})},
$$

which together with (4.10) concludes the proof of (4.1). □

4.2. $L^1_T(\mathcal{B}^{s,0})$ estimate of $\boldsymbol{f}(Y,q)$ given by (2.22).

Proposition 4.2. *Let (Y,q) be a smooth enough solution of (2.20) (or equivalently of (2.21)-(2.22)) on $[0,T]$ with the initial data (Y_0,Y_1) satisfying $\det(I+\nabla Y_0)=1$ and $\nabla_{Y_0}\cdot Y_1=0$. If we assume moreover that*

$$
\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\le c_0\quad\text{and}\quad \big\langle \|Y\|_{L^\infty_T(\dot B^{s+\frac54})}+\|Y\|_{L^\infty_T(\dot B^{s+\frac32})}\big\rangle_{s>1}\le 1\tag{4.11}
$$

for some c_0 sufficiently small. Then there holds

$$
\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\lesssim \|Y_t\|_{L^2_T(\dot B^{\frac94})}\|Y_t\|_{L^2_T(\dot B^{s+\frac14})}+\|\partial_3 Y\|_{L^2_T(\dot B^{\frac94})}\|\partial_3 Y\|_{L^2_T(\dot B^{s+\frac14})}\tag{4.12}
$$

if $0<s\le 1$, and

$$
\begin{aligned}
\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\lesssim&\|Y_t\|^2_{L^2_T(\dot B^{\frac54})}+\|Y_t\|^2_{L^2_T(\dot B^{\frac52})}+\|Y_t\|^2_{L^2_T(\dot B^{s+\frac54})}\\
&+\|\partial_3 Y\|^2_{L^2_T(\dot B^{\frac32})}+\|\partial_3 Y\|^2_{L^2_T(\dot B^{\frac94})}+\|\partial_3 Y\|^2_{L^2_T(\dot B^{s+\frac14})}\quad\text{if}\ \ s>1.
\end{aligned}\tag{4.13}
$$

Here and in all that follows, $A_s=B_s+\langle C_s\rangle_{s>s_0}$ means $A_s=B_s$ if $s\le s_0$ and $A_s=B_s+C_s$ if $s>s_0$.

Proof. By virtue of (2.20), we get, by taking ∂_t to $\nabla_Y\cdot Y_t=0$, that

$$
\nabla_Y\cdot Y_{tt}=-\partial_t\mathcal{A}_Y^T\nabla\cdot Y_t.\tag{4.14}
$$

Note that for c_0 in (4.11) being so small that

$$
\|\nabla Y\|_{L^\infty_T(L^\infty)}\le C\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\le Cc_0\le\frac12,
$$

$X(t,y)$ determined by (2.15) has a smooth inverse map $X^{-1}(t,x)$ with $X(t,X^{-1}(t,x))=x$ and $X^{-1}(t,X(t,y))=y$. Moreover, as $\det(I+\nabla Y_0)=1$, we deduce from (2.23) that $\det(I+\nabla Y)=1$, which together with $\nabla_Y\cdot Y_t=0$ ensures that

$$
\nabla_Y\cdot(\nabla_Y\cdot\nabla_Y Y_t)=[\nabla_x\cdot\Delta_x(Y_t\circ X^{-1}(t,x))]\circ X(t,y)=\nabla_Y\cdot\nabla_Y(\nabla_Y\cdot Y_t)=0,
$$

from which and (4.14), we get, by taking $\nabla_Y\cdot$ to the first equation of (2.20), that

$$\nabla_Y\cdot\nabla_Y q=\partial_t\mathcal{A}_Y^T\nabla\cdot Y_t+\nabla_Y\cdot\partial_3^2Y,$$

or equivalently

$$\begin{aligned}\Delta q=&-(\nabla_Y-\nabla)\cdot\nabla_Y q-\nabla\cdot(\nabla_Y-\nabla)q+\partial_t\mathcal{A}_Y^T\nabla\cdot Y_t+\nabla_Y\cdot\partial_3^2Y\\=&-\nabla\cdot\big((\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla q\big)-\nabla\cdot\big((\mathcal{A}_Y^T-I)\nabla q\big)+\nabla\cdot(\partial_t\mathcal{A}_YY_t)+\nabla_Y\cdot\partial_3^2Y,\end{aligned}\tag{4.15}$$

from which, Lemma 3.1 and Definition 3.2, we infer

$$\begin{aligned}\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\leq&\|(\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}+\|(\mathcal{A}_Y^T-I)\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\\&+\|\partial_t\mathcal{A}_YY_t\|_{L^1_T(\mathcal{B}^{s,0})}+\|\nabla_Y\cdot\partial_3^2Y\|_{L^1_T(\mathcal{B}^{s-1,0})}.\end{aligned}\tag{4.16}$$

Applying (3.10) and (2.17) gives for $0<s\leq1$

$$\begin{aligned}\|(\mathcal{A}_Y^T-I)\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}&+\|(\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\\&\lesssim\big(\|\mathcal{A}_Y^T-I\|_{L^\infty_T(\dot B^{\frac32})}+\|(\mathcal{A}_Y-I)\mathcal{A}_Y^T\|_{L^\infty_T(\dot B^{\frac32})}\big)\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\\&\lesssim\big(1+\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\big)^3\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})},\end{aligned}$$

and

$$\begin{aligned}\|\partial_t\mathcal{A}_YY_t\|_{L^1_T(\mathcal{B}^{s,0})}\lesssim&\|\partial_t\mathcal{A}_Y\|_{L^2_T(\dot B^{\frac54})}\|Y_t\|_{L^2_T(\dot B^{s+\frac14})}\\\lesssim&\big(1+\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\big)\|\nabla Y_t\|_{L^2_T(\dot B^{\frac54})}\|Y_t\|_{L^2_T(\dot B^{s+\frac14})}.\end{aligned}$$

While thanks to (2.17), (2.21) and (2.22), a tedious yet interesting calculation shows that

$$\nabla_Y\cdot\partial_3^2Y=\nabla\cdot\big((\mathcal{A}_Y-I)\partial_3^2Y\big)+\partial_3^2\rho(Y)=Q(\nabla\partial_3Y,\nabla\partial_3Y,\nabla Y),\tag{4.17}$$

where $Q(\nabla\partial_3Y,\nabla\partial_3Y,\nabla Y)$ is a linear combination of quadratic terms like $\partial_3\partial_iY^j\partial_3\partial_kY^l$ and cubic terms like $\partial_pY^q\partial_3\partial_iY^j\partial_3\partial_kY^l$. Then applying (3.10) to (4.17) ensures that for $0<s\leq2$

$$\begin{aligned}\|\nabla_Y\cdot\partial_3^2Y\|_{L^1_T(\mathcal{B}^{s-1,0})}\lesssim&\|(I+\nabla Y)\partial_3\nabla Y\|_{L^2_T(\dot B^{\frac54})}\|\partial_3\nabla Y\|_{L^2_T(\dot B^{s-\frac34})}\\\lesssim&\big(1+\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\big)\|\partial_3\nabla Y\|_{L^2_T(\dot B^{\frac54})}\|\partial_3Y\|_{L^2_T(\dot B^{s+\frac14})}.\end{aligned}\tag{4.18}$$

Thus for $0<s\leq1$, resuming the above estimates into (4.16) gives rise to

$$\begin{aligned}\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\lesssim&\big(1+\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\big)^3\Big(\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\\&+\|Y_t\|_{L^2_T(\dot B^{\frac94})}\|Y_t\|_{L^2_T(\dot B^{s+\frac14})}+\|\partial_3Y\|_{L^2_T(\dot B^{\frac94})}\|\partial_3Y\|_{L^2_T(\dot B^{s+\frac14})}\Big),\end{aligned}$$

which along with (4.11) implies (4.12).

On the other hand, we get, by applying (3.11) and (3.12), that for $s>1$

$$\begin{aligned}&\|(\mathcal{A}_Y^T-I)\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}+\|(\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\\&\lesssim\Big(\|\mathcal{A}_Y^T-I\|_{L^\infty_T(\dot B^{\frac32})}+\|(\mathcal{A}_Y-I)\mathcal{A}_Y^T\|_{L^\infty_T(\dot B^{\frac32})}\Big)\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\\&\quad+\Big(\|\mathcal{A}_Y^T-I\|_{L^\infty_T(\dot B^{s+\frac12})}+\|(\mathcal{A}_Y-I)\mathcal{A}_Y^T\|_{L^\infty_T(\dot B^{s+\frac12})}\Big)\|\nabla q\|_{L^1_T(\mathcal{B}^{1,0})}\\&\lesssim\big(1+\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\big)^3\Big(\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}+\|\nabla Y\|_{L^\infty_T(\dot B^{s+\frac12})}\|\nabla q\|_{L^1_T(\mathcal{B}^{1,0})}\Big),\end{aligned}\tag{4.19}$$

and

$$
\begin{aligned}
\|\partial_t\mathcal{A}_YY_t\|_{L^1_T(\mathcal{B}^{s,0})} \lesssim & \|\partial_t\mathcal{A}_Y\|_{L^2_T(\dot{B}^{\frac{5}{4}})}\|Y_t\|_{L^2_T(\dot{B}^{s+\frac{1}{4}})} + \|\partial_t\mathcal{A}_Y\|_{L^2_T(\dot{B}^{s+\frac{1}{4}})}\|Y_t\|_{L^2_T(\dot{B}^{\frac{5}{4}})} \\
\lesssim & \big(1+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}+\|\nabla Y\|_{L^\infty_T(\dot{B}^{s+\frac{1}{4}})}\big)\Big(\|\nabla Y_t\|_{L^2_T(\dot{B}^{\frac{5}{4}})}\|Y_t\|_{L^2_T(\dot{B}^{s+\frac{1}{4}})} \\
& + \big(\|\nabla Y_t\|_{L^2_T(\dot{B}^{\frac{3}{2}})}+\|\nabla Y_t\|_{L^2_T(\dot{B}^{s+\frac{1}{4}})}\big)\|Y_t\|_{L^2_T(\dot{B}^{\frac{5}{4}})}\Big).
\end{aligned}
\tag{4.20}
$$

Moreover, applying (3.12) to (4.17) leads to

$$
\begin{aligned}
\|\nabla_Y\cdot\partial_3^2Y\|_{L^1_T(\mathcal{B}^{s-1,0})} \lesssim & \|(I+\nabla Y)\partial_3\nabla Y\|_{L^2_T(\dot{B}^{\frac{5}{4}})}\|\partial_3\nabla Y\|_{L^2_T(\dot{B}^{s-\frac{3}{4}})} \\
& + \|(I+\nabla Y)\partial_3\nabla Y\|_{L^2_T(\dot{B}^{s-\frac{3}{4}})}\|\partial_3\nabla Y\|_{L^2_T(\dot{B}^{\frac{5}{4}})},
\end{aligned}
$$

however, by applying Bony's decomposition (3.8), one has

$$
\begin{aligned}
\|(I+\nabla Y)\partial_3\nabla Y\|_{L^2_T(\dot{B}^{s-\frac{3}{4}})} \lesssim & \big(1+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}\big)\|\partial_3\nabla Y\|_{L^2_T(\dot{B}^{s-\frac{3}{4}})} \\
& + \|\nabla Y\|_{L^\infty_T(\dot{B}^{s+\frac{1}{4}})}\|\partial_3\nabla Y\|_{L^2_T(\dot{B}^{\frac{1}{2}})},
\end{aligned}
$$

so that for $s>2$, we achieve

$$
\begin{aligned}
\|\nabla_Y\cdot\partial_3^2Y\|_{L^1_T(\mathcal{B}^{s-1,0})} \lesssim & \Big(1+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}+\|Y\|_{L^\infty_T(\dot{B}^{s+\frac{5}{4}})}\Big) \\
& \times\Big(\|\partial_3 Y\|^2_{L^2_T(\dot{B}^{\frac{9}{4}})}+\|\partial_3 Y\|^2_{L^2_T(\dot{B}^{\frac{3}{2}})}+\|\partial_3 Y\|^2_{L^2_T(\dot{B}^{s+\frac{1}{4}})}\Big).
\end{aligned}
\tag{4.21}
$$

Whence plugging (4.19), (4.20), (4.18) and (4.21) into (4.16), we obtain

$$
\begin{aligned}
\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})} \lesssim & \big(1+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}\big)^3\Big(\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})} \\
& + \|Y\|_{L^\infty_T(\dot{B}^{s+\frac{3}{2}})}\|\nabla q\|_{L^1_T(\mathcal{B}^{1,0})}\Big) + \big(1+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}+\|Y\|_{L^\infty_T(\dot{B}^{s+\frac{5}{4}})}\big) \\
& \times\Big(\|Y_t\|^2_{L^2_T(\dot{B}^{\frac{5}{4}})}+\|Y_t\|^2_{L^2_T(\dot{B}^{\frac{5}{2}})}+\|Y_t\|^2_{L^2_T(\dot{B}^{s+\frac{5}{4}})} \\
& + \|\partial_3 Y\|^2_{L^2_T(\dot{B}^{\frac{3}{2}})}+\|\partial_3 Y\|^2_{L^2_T(\dot{B}^{\frac{9}{4}})}+\|\partial_3 Y\|^2_{L^2_T(\dot{B}^{s+\frac{1}{4}})}\Big) \quad \text{for} \quad s>1,
\end{aligned}
$$

which together with (4.11) and (4.12) implies (4.13). This completes the proof of Proposition 4.2. □

Corollary 4.1. *Under the assumption of Proposition 4.2, one has*

$$
\|\nabla q\|_{L^1_T(\dot{B}^{\frac{1}{2}})} \lesssim \|Y_t\|^2_{L^2_T(\dot{B}^{\frac{3}{2}})}+\|\partial_3 Y\|^2_{L^2_T(\dot{B}^{\frac{3}{2}})}.
\tag{4.22}
$$

Proof. We first deduce from (4.15) that

$$
\begin{aligned}
\|\nabla q\|_{L^1_T(\dot{B}^{\frac{1}{2}})} \le & \|(\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla q\|_{L^1_T(\dot{B}^{\frac{1}{2}})}+\|(\mathcal{A}_Y^T-I)\nabla q\|_{L^1_T(\dot{B}^{\frac{1}{2}})} \\
& + \|\partial_t\mathcal{A}_YY_t\|_{L^1_T(\dot{B}^{\frac{1}{2}})}+\|\nabla_Y\cdot\partial_3^2Y\|_{L^1_T(\dot{B}^{-\frac{1}{2}})}.
\end{aligned}
\tag{4.23}
$$

It follows from product laws in Besov space (see [2] for instance) that

$$
\begin{aligned}
& \|(\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla q\|_{L^1_T(\dot{B}^{\frac{1}{2}})}+\|(\mathcal{A}_Y^T-I)\nabla q\|_{L^1_T(\dot{B}^{\frac{1}{2}})} \\
& \qquad \lesssim \big(1+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}\big)^3\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}\|\nabla q\|_{L^1_T(\dot{B}^{\frac{1}{2}})},
\end{aligned}
$$

and

$$\|\partial_t\mathcal{A}_Y Y_t\|_{L^1_T(\dot{B}^{\frac12})}\lesssim\|\partial_t\mathcal{A}_Y\|_{L^2_T(\dot{B}^{\frac12})}\|Y_t\|_{L^2_T(\dot{B}^{\frac32})}\lesssim\big(1+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac32})}\big)\|Y_t\|^2_{L^2_T(\dot{B}^{\frac32})}.$$

Along the same line, we deduce from (4.17) that

$$\begin{aligned}\|\nabla_Y\cdot\partial_3^2Y\|_{L^1_T(\dot{B}^{-\frac12})}&\lesssim\|Q(\nabla\partial_3Y,\nabla\partial_3Y,\nabla Y)\|_{L^1_T(\dot{B}^{-\frac12})}\\&\lesssim\|(I+\nabla Y)\nabla\partial_3Y\|_{L^2_T(\dot{B}^{\frac12})}\|\nabla\partial_3Y\|_{L^2_T(\dot{B}^{\frac12})}\lesssim\big(1+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac32})}\big)\|\partial_3Y\|^2_{L^2_T(\dot{B}^{\frac32})}.\end{aligned}$$

Resuming the above estimates into (4.23) leads to (4.22). ☐

Proposition 4.3. *Let $s>1$, and $\boldsymbol{f}(Y,q)$ be given by (2.22). Then under the assumptions of Proposition 4.2 and*

$$\|Y\|_{L^\infty_T(\mathcal{B}^{s+2,0})}\leq 1,\tag{4.24}$$

one has

$$\begin{aligned}&\|\boldsymbol{f}(Y,q)\|_{L^1_T(\mathcal{B}^{\frac12,0})}+\|\boldsymbol{f}(Y,q)\|_{L^1_T(\mathcal{B}^{s,0})}\lesssim\|\nabla q\|_{L^1_T(\mathcal{B}^{\frac12,0})}+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac32})}\|Y_t\|_{L^1_T(\mathcal{B}^{s+2,0})}\\&\quad+\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}+\|Y_t\|_{L^1_T(\dot{B}^{\frac52})}\Big(\|Y\|_{L^\infty_T(\mathcal{B}^{s+2,0})}+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac32})}\Big).\end{aligned}\tag{4.25}$$

Proof. Thanks to (2.22), we split $\boldsymbol{f}(Y,q)$ as follows:

$$\begin{aligned}&\boldsymbol{f}(Y,q)=\bar{\boldsymbol{f}}(Y)+\tilde{\boldsymbol{f}}(Y,q),\quad\text{with}\\&\bar{\boldsymbol{f}}(Y)\stackrel{\text{def}}{=}(\nabla_Y\cdot\nabla_Y-\Delta)Y_t\quad\text{and}\quad\tilde{\boldsymbol{f}}(Y,q)\stackrel{\text{def}}{=}-\nabla_Yq.\end{aligned}\tag{4.26}$$

As $\tilde{\boldsymbol{f}}(Y,q)=-(\mathcal{A}^T-I)\nabla q-\nabla q$, by virtue of (2.17), we deduce from (3.10) and (4.11) that

$$\begin{aligned}\|\tilde{\boldsymbol{f}}(Y,q)\|_{L^1_T(\mathcal{B}^{\frac12,0})}&\lesssim\big(1+\|\mathcal{A}^T-I\|_{L^\infty_T(\dot{B}^{\frac32})}\big)\|\nabla q\|_{L^1_T(\mathcal{B}^{\frac12,0})}\\&\lesssim\big(1+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac32})}\big)^2\|\nabla q\|_{L^1_T(\mathcal{B}^{\frac12,0})}\lesssim\|\nabla q\|_{L^1_T(\mathcal{B}^{\frac12,0})},\end{aligned}\tag{4.27}$$

and for $s>1$, we infer from (3.11) and (4.11) that

$$\begin{aligned}\|\tilde{\boldsymbol{f}}(Y,q)\|_{L^1_T(\mathcal{B}^{s,0})}\lesssim&\|\mathcal{A}^T-I\|_{L^\infty_T(\dot{B}^{\frac32})}\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\\&+\|\mathcal{A}^T-I\|_{L^\infty_T(\dot{B}^{s+\frac12})}\|\nabla q\|_{L^1_T(\mathcal{B}^{1,0})}+\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\\\lesssim&\big(1+\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac32})}\big)\Big(\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac32})}\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\\&+\|\nabla Y\|_{L^\infty_T(\dot{B}^{s+\frac12})}\|\nabla q\|_{L^1_T(\mathcal{B}^{1,0})}\Big)+\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}\\\lesssim&\|\nabla q\|_{L^1_T(\mathcal{B}^{\frac12,0})}+\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})},\end{aligned}\tag{4.28}$$

where in the last step, we used the trivial fact that

$$\|a\|_{\mathcal{B}^{s,\delta}}\lesssim\|a\|_{\mathcal{B}^{s_1,\delta}}+\|a\|_{\mathcal{B}^{s_2,\delta}}\quad\text{for any}\quad s\in[s_1,s_2].\tag{4.29}$$

On the other hand, notice that

$$\begin{aligned}\bar{\boldsymbol{f}}^i(Y)&=(\nabla_Y-\nabla)\cdot\nabla_YY_t^i+\nabla\cdot(\nabla_Y-\nabla)Y_t^i\\&=\nabla\cdot\big[(\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla Y_t^i+(\mathcal{A}_Y^T-I)\nabla Y_t^i\big],\end{aligned}\tag{4.30}$$

which leads to

$$\|\bar{\boldsymbol{f}}(Y)\|_{L^1_T(\mathcal{B}^{s,0})}\lesssim\|(\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla Y_t\|_{L^1_T(\mathcal{B}^{s+1,0})}+\|(\mathcal{A}_Y^T-I)\nabla Y_t\|_{L^1_T(\mathcal{B}^{s+1,0})}.\tag{4.31}$$

Whereas according to (2.17), we get, by applying (3.11), that

$$\begin{aligned}
&\|(\mathcal{A}_Y^T-I)\nabla Y_t\|_{L^1_T(\mathcal{B}^{s+1,0})}\\
&\quad\lesssim\|\mathcal{A}_Y^T-I\|_{L^\infty_T(\dot B^{\frac32})}\|\nabla Y_t\|_{L^1_T(\mathcal{B}^{s+1,0})}+\|\mathcal{A}_Y^T-I\|_{L^\infty_T(\mathcal{B}^{s+1,0})}\|\nabla Y_t\|_{L^1_T(\dot B^{\frac32})}\\
&\quad\lesssim\big(1+\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\big)\Big(\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\|Y_t\|_{L^1_T(\mathcal{B}^{s+2,0})}+\|Y_t\|_{L^1_T(\dot B^{\frac52})}\|\nabla Y\|_{L^\infty_T(\mathcal{B}^{s+1,0})}\Big).
\end{aligned}$$

Along the same line, and thanks to (4.11) and (4.24), we have

$$\begin{aligned}
\|(\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla Y_t\|_{L^1_T(\mathcal{B}^{s+1,0})}\lesssim{}&\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\|Y_t\|_{L^1_T(\mathcal{B}^{s+2,0})}\\
&+\|\nabla Y_t\|_{L^1_T(\dot B^{\frac32})}\Big(\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}+\|\nabla Y\|_{L^\infty_T(\mathcal{B}^{s+1,0})}\Big).
\end{aligned}$$

Resuming the above two estimates into (4.31) gives rise to

$$\begin{aligned}
\|\bar{\boldsymbol{f}}(Y)\|_{L^1_T(\mathcal{B}^{s,0})}\lesssim{}&\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\|Y_t\|_{L^1_T(\mathcal{B}^{s+2,0})}\\
&+\|Y_t\|_{L^1_T(\dot B^{\frac52})}\Big(\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}+\|Y\|_{L^\infty_T(\mathcal{B}^{s+2,0})}\Big),
\end{aligned}$$

which together with (4.27) and (4.28) ensures (4.25). This concludes the proof of Proposition 4.3. □

4.3. $L^1_T(\mathcal{B}^{s+2,0}(\mathbb{R}^3))$ estimate of Y_t for $s=\frac12$ and $s>1$.

Proposition 4.4. *Let $s>1$, then under assumptions of Proposition 4.2 and*

$$\|Y\|_{L^\infty_T(\mathcal{B}^{s+2,0})}\leq c_0, \tag{4.32}$$

for some c_0 sufficiently small, we have

$$\begin{aligned}
&\|Y_t\|_{\widetilde L^\infty_T(\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0})}+\|\partial_3 Y\|_{\widetilde L^\infty_T(\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0})}+\|Y\|_{\widetilde L^\infty_T(\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s+2,0})}\\
&\quad+\|Y_t\|_{L^1_T(\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s+2,0})}+\|\partial_3 Y\|_{\widetilde L^2_T(\mathcal{B}^{\frac32,0}\cap\mathcal{B}^{s+1,0})}\\
&\lesssim\|Y_1\|_{\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0}}+\|\partial_3 Y_0\|_{\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0}}+\|Y_0\|_{\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s+2,0}}+\|Y_t\|^2_{L^2_T(\dot B^{\frac34})}\\
&\quad+\|Y_t\|^2_{L^2_T(\dot B^{\frac52})}+\|Y_t\|^2_{L^2_T(\dot B^{s+\frac54})}+\|\partial_3 Y\|^2_{L^2_T(\dot B^{\frac34})}+\|\partial_3 Y\|^2_{L^2_T(\dot B^{\frac94})}\\
&\quad+\|\partial_3 Y\|^2_{L^2_T(\dot B^{s+\frac14})}+\|Y\|^2_{L^\infty_T(\mathcal{B}^{s+2,0})}.
\end{aligned}\tag{4.33}$$

Proof. Thanks to Propositions 4.1 and 4.3, we conclude that for $s>1$,

$$\begin{aligned}
&\|Y_t\|_{\widetilde L^\infty_T(\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0})}+\|\partial_3 Y\|_{\widetilde L^\infty_T(\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0})}+\|Y\|_{\widetilde L^\infty_T(\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s+2,0})}\\
&\quad+\|Y_t\|_{L^1_T(\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s+2,0})}+\|\partial_3 Y\|_{\widetilde L^2_T(\mathcal{B}^{\frac32,0}\cap\mathcal{B}^{s+1,0})}\\
&\lesssim\|Y_1\|_{\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0}}+\|\partial_3 Y_0\|_{\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0}}+\|Y_0\|_{\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s+2,0}}+\|\nabla q\|_{L^1_T(\mathcal{B}^{\frac12,0})}\\
&\quad+\|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}+\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\|Y_t\|_{L^1_T(\mathcal{B}^{s+2,0})}\\
&\quad+\|Y_t\|_{L^1_T(\dot B^{\frac52})}\Big(\|Y\|_{L^\infty_T(\mathcal{B}^{s+2,0})}+\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\Big),
\end{aligned}$$

which together with (4.11), (4.32) and the fact that $\|Y_t\|_{L^1_T(\dot B^{\frac52})} \lesssim \|Y_t\|_{L^1_T(\mathcal{B}^{\frac52,0})}$ ensures that

$$\begin{aligned}
&\|Y_t\|_{\widetilde{L}^\infty_T(\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0})} + \|\partial_3 Y\|_{\widetilde{L}^\infty_T(\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0})} + \|Y\|_{\widetilde{L}^\infty_T(\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s+2,0})}\\
&\quad + \|Y_t\|_{L^1_T(\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s+2,0})} + \|\partial_3 Y\|_{\widetilde{L}^2_T(\mathcal{B}^{\frac32,0}\cap\mathcal{B}^{s+1,0})}\\
&\lesssim \|Y_1\|_{\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0}} + \|\partial_3 Y_0\|_{\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s,0}} + \|Y_0\|_{\mathcal{B}^{\frac52,0}\cap\mathcal{B}^{s+2,0}}\\
&\quad + \|\nabla q\|_{L^1_T(\mathcal{B}^{\frac12,0})} + \|\nabla q\|_{L^1_T(\mathcal{B}^{s,0})}.
\end{aligned}$$

Hence applying Proposition 4.2 gives rise to (4.33). This complete the proof of Proposition 4.4. ◻

5. The proof of Theorem 2.2

5.1. *A priori* **estimate of** (2.20)**.** The goal of this subsection is to present the *a priori* energy estimate to smooth enough solutions of (2.20).

Lemma 5.1. *Let Y be a smooth enough solution of* (2.21)-(2.22) *(or equivalently* (2.20)*) on $[0,T]$. Then there holds*

$$\begin{aligned}
&\|\Delta_j Y_t\|^2_{L^\infty_T(L^2)} + \|\nabla\Delta_j Y_t\|^2_{L^\infty_T(L^2)} + \|\partial_3\Delta_j Y\|^2_{L^\infty_T(L^2)} + \|\Delta\Delta_j Y\|^2_{L^\infty_T(L^2)}\\
&\quad + \|\nabla\Delta_j Y_t\|^2_{L^2_T(L^2)} + \|\Delta\Delta_j Y_t\|^2_{L^2_T(L^2)} + \|\partial_3\nabla\Delta_j Y\|^2_{L^2_T(L^2)}\\
&\lesssim \|\Delta_j Y_1\|^2_{L^2} + \|\nabla\Delta_j Y_1\|^2_{L^2} + \|\partial_3\Delta_j Y_0\|^2_{L^2} + \|\Delta\Delta_j Y_0\|^2_{L^2}\\
&\quad + \Bigl|\int_0^T \bigl(\Delta_j \boldsymbol{f} \mid \Delta_j Y_t - \frac14\Delta\Delta_j Y - \Delta\Delta_j Y_t\bigr)\,dt\Bigr|.
\end{aligned} \tag{5.1}$$

Proof. Applying Δ_j to (2.21) gives

$$\Delta_j Y_{tt} - \Delta\Delta_j Y_t - \partial_3^2\Delta_j Y = \Delta_j \boldsymbol{f}. \tag{5.2}$$

Taking the L^2 inner product of (5.2) with $\Delta_j Y_t - \frac14\Delta\Delta_j Y - \Delta\Delta_j Y_t$, we get, by a similar derivation of (4.5), that

$$\begin{aligned}
&\frac{d}{dt}\Bigl\{\frac12\bigl(\|\Delta_j Y_t\|^2_{L^2} + \|\nabla\Delta_j Y_t\|^2_{L^2} + \|\partial_3\Delta_j Y\|^2_{L^2} + \|\partial_3\nabla\Delta_j Y\|^2_{L^2} + \frac14\|\Delta\Delta_j Y\|^2_{L^2}\bigr)\\
&\quad - \frac14(\Delta_j Y_t \mid \Delta\Delta_j Y)\Bigr\} + \frac34\|\nabla\Delta_j Y_t\|^2_{L^2} + \|\Delta\Delta_j Y_t\|^2_{L^2} + \frac14\|\partial_3\nabla\Delta_j Y\|^2_{L^2}\\
&= \bigl(\Delta_j \boldsymbol{f} \mid \Delta_j Y_t - \frac14\Delta\Delta_j Y - \Delta\Delta_j Y_t\bigr).
\end{aligned} \tag{5.3}$$

However, as

$$\begin{aligned}
&\frac12\bigl(\|\Delta_j Y_t\|^2_{L^2} + \|\nabla\Delta_j Y_t\|^2_{L^2} + \|\partial_3\Delta_j Y\|^2_{L^2} + \|\partial_3\nabla\Delta_j Y\|^2_{L^2} + \frac14\|\Delta\Delta_j Y\|^2_{L^2}\bigr)\\
&\quad - \frac14(\Delta_j Y_t \mid \Delta\Delta_j Y) \sim \|\Delta_j Y_t(t)\|^2_{L^2} + \|\nabla\Delta_j Y_t\|^2_{L^2} + \|\partial_3\Delta_j Y(t)\|^2_{L^2} + \|\Delta\Delta_j Y(t)\|^2_{L^2},
\end{aligned}$$

by integrating (5.3) over $[0,T]$, we obtain (5.1). ◻

To deal with the last line of (5.1), we need to estimate the $\boldsymbol{f}(Y,q)$ given by (2.22). Toward this, we first deal with the the pressure term in (2.21).

Lemma 5.2. *Under the assumptions of Proposition 4.2, for any $s>-\frac{1}{2}$, one has*

$$\begin{aligned}\|\nabla q(t)\|_{\dot H^s} \lesssim&\|Y(t)\|_{\dot H^{s+2}}\|\nabla q(t)\|_{\dot B^{\frac12}}+\|Y_t(t)\|_{\dot B^{\frac32}}\|Y_t(t)\|_{\dot H^{s+1}}\\&+\|Y(t)\|_{\dot H^{s+2}}\big(\|Y_t(t)\|^2_{\dot B^{\frac32}}+\|\partial_3 Y(t)\|^2_{\dot B^{\frac32}}\big)+\|\partial_3 Y(t)\|_{\dot B^{\frac32}}\|\partial_3 Y(t)\|_{\dot H^{s+1}},\end{aligned}\tag{5.4}$$

for all $t\in[0,T]$.

Proof. For any $t\in[0,T]$, we deduce from (4.15) that

$$\begin{aligned}\|\nabla q(t)\|_{\dot H^s}\leq&\|((\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla q)(t)\|_{\dot H^s}+\|((\mathcal{A}_Y^T-I)\nabla q)(t)\|_{\dot H^s}\\&+\|(\partial_t\mathcal{A}_Y Y_t)(t)\|_{\dot H^s}+\|(\nabla_Y\cdot\partial_3^2 Y)(t)\|_{\dot H^{s-1}}.\end{aligned}\tag{5.5}$$

By virtue of (2.17), we get, by using Bony's decomposition (3.8), that for any $s>-\frac12$,

$$\begin{aligned}&\|((\mathcal{A}_Y-I)\mathcal{A}_Y^T\nabla q)(t)\|_{\dot H^s}+\|((\mathcal{A}_Y^T-I)\nabla q)(t)\|_{\dot H^s}\\&\quad\lesssim\big(1+\|\nabla Y(t)\|_{\dot B^{\frac32}}\big)^3\Big(\|\nabla Y(t)\|_{\dot B^{\frac32}}\|\nabla q(t)\|_{\dot H^s}+\|\nabla Y(t)\|_{\dot H^{s+1}}\|\nabla q(t)\|_{\dot B^{\frac12}}\Big),\end{aligned}$$

and

$$\|(\partial_t\mathcal{A}_Y Y_t)(t)\|_{\dot H^s}\lesssim\big(1+\|\nabla Y(t)\|_{\dot B^{\frac32}}\big)\|Y_t(t)\|_{\dot B^{\frac32}}\|Y_t(t)\|_{\dot H^{s+1}}+\|\nabla Y(t)\|_{\dot H^{s+1}}\|Y_t(t)\|^2_{\dot B^{\frac32}}.$$

Along the same line, due to (4.17), we obtain for any $s>-\frac12$,

$$\begin{aligned}&\|(\nabla_Y\cdot\partial_3^2 Y)(t)\|_{\dot H^{s-1}}\lesssim\|Q(\nabla\partial_3 Y,\nabla\partial_3 Y,\nabla Y)(t)\|_{\dot H^{s-1}}\\&\quad\lesssim\|((I+\nabla Y)\partial_3\nabla Y)(t)\|_{\dot B^{\frac12}}\|\partial_3 Y(t)\|_{\dot H^{s+1}}+\|((I+\nabla Y)\partial_3\nabla Y)(t)\|_{\dot H^s}\|\partial_3 Y(t)\|_{\dot B^{\frac32}}\\&\quad\lesssim\big(1+\|\nabla Y(t)\|_{\dot B^{\frac32}}\big)\|\partial_3 Y(t)\|_{\dot B^{\frac32}}\|\partial_3 Y(t)\|_{\dot H^{s+1}}+\|Y(t)\|_{\dot H^{s+2}}\|\partial_3 Y(t)\|^2_{\dot B^{\frac32}}.\end{aligned}$$

Resuming the above estimates into (5.5) and using (4.11) ensures that for any $s>-\frac12$,

$$\begin{aligned}\|\nabla q(t)\|_{\dot H^s}\lesssim&\|\nabla Y(t)\|_{\dot B^{\frac32}}\|\nabla q(t)\|_{\dot H^s}+\|\nabla Y(t)\|_{\dot H^{s+1}}\|\nabla q(t)\|_{\dot B^{\frac12}}+\|Y_t(t)\|_{\dot B^{\frac32}}\|Y_t(t)\|_{\dot H^{s+1}}\\&+\|Y(t)\|_{\dot H^{s+2}}\big(\|Y_t(t)\|^2_{\dot B^{\frac32}}+\|\partial_3 Y(t)\|^2_{\dot B^{\frac32}}\big)+\|\partial_3 Y(t)\|_{\dot B^{\frac32}}\|\partial_3 Y(t)\|_{\dot H^{s+1}},\end{aligned}$$

for any $t\in[0,T]$, which together (4.11) leads to (5.4). ☐

Lemma 5.3. *Under the assumptions of Proposition 4.2, for any $s>-\frac12$, we have*

$$\begin{aligned}\|\boldsymbol{f}(Y,q)\|_{L^2_T(\dot H^s)}\lesssim&\|Y\|_{L^\infty_T(\dot H^{s+2})}\Big(\|\nabla q\|_{L^2_T(\dot B^{\frac12})}+\|Y_t\|_{L^2_T(\dot B^{\frac52})}+\|Y_t\|_{L^\infty_T(\dot B^{\frac32})}\|Y_t\|_{L^2_T(\dot B^{\frac32})}\\&+\|\partial_3 Y\|_{L^\infty_T(\dot B^{\frac32})}\|\partial_3 Y\|_{L^2_T(\dot B^{\frac32})}\Big)+\|Y_t\|_{L^\infty_T(\dot B^{\frac32})}\|Y_t\|_{L^2_T(\dot H^{s+1})}\\&+\|\partial_3 Y\|_{L^\infty_T(\dot B^{\frac32})}\|\partial_3 Y\|_{L^2_T(\dot H^{s+1})}+\|\nabla Y\|_{L^\infty_T(\dot B^{\frac32})}\|Y_t\|_{L^2_T(\dot H^{s+2})}\\&,\end{aligned}\tag{5.6}$$

and for $-\frac12<s\leq\frac12$,

$$\begin{aligned}\|\boldsymbol{f}(Y,q)\|_{L^1_T(\dot H^s)}\lesssim&\|Y\|_{L^\infty_T(\dot H^{s+2})}\Big(\|\nabla q\|_{L^1_T(\dot B^{\frac12})}+\|Y_t\|_{L^1_T(\dot B^{\frac52})}+\|Y_t\|^2_{L^2_T(\dot B^{\frac32})}\\&+\|\partial_3 Y\|^2_{L^2_T(\dot B^{\frac32})}\Big)+\|Y_t\|_{L^2_T(\dot B^{\frac32})}\|Y_t\|_{L^2_T(\dot H^{s+1})}+\|\partial_3 Y\|_{L^2_T(\dot B^{\frac32}_{2,1})}\|\partial_3 Y\|_{L^2_T(\dot H^{s+1})}.\end{aligned}\tag{5.7}$$

Proof. According to (4.26), we split the estimate of $\boldsymbol{f}(Y,q)$ into that of $\tilde{\boldsymbol{f}}(Y,q)$ and $\bar{\boldsymbol{f}}(Y)$.

- **Estimates on $\tilde{\boldsymbol{f}}(Y,q)=-\nabla_Y q$.**

Thanks to (2.17), we get, by using product laws in Besov spaces ([2]), that $s > -\frac{1}{2}$,

$$\|\tilde{\boldsymbol{f}}(Y,q)(t)\|_{\dot{H}^s} \lesssim \big(1 + \|\nabla Y(t)\|_{\dot{B}^{\frac{3}{2}}}\big)\big(\|\nabla Y(t)\|_{\dot{B}^{\frac{3}{2}}}\|\nabla q(t)\|_{\dot{H}^s} + \|\nabla Y(t)\|_{\dot{H}^{s+1}}\|\nabla q(t)\|_{\dot{B}^{\frac{1}{2}}}\big) + \|\nabla q(t)\|_{\dot{H}^s}$$

which along with (4.11) implies that for any $s > -\frac{1}{2}$,

$$\|\tilde{\boldsymbol{f}}(Y,q)(t)\|_{\dot{H}^s} \lesssim \|\nabla q(t)\|_{\dot{H}^s} + \|Y(t)\|_{\dot{H}^{s+2}}\|\nabla q(t)\|_{\dot{B}^{\frac{1}{2}}}. \tag{5.8}$$

- **Estimates on $\bar{\boldsymbol{f}}(Y) = (\nabla_Y \cdot \nabla_Y - \Delta)Y_t$.**

It follows from (4.30) that

$$\|\bar{\boldsymbol{f}}(Y)(t)\|_{\dot{H}^s} \lesssim \|[(\mathcal{A}_Y - I)\mathcal{A}_Y^T\nabla Y_t](t)\|_{\dot{H}^{s+1}} + \|[(\mathcal{A}_Y^T - I)\nabla Y_t](t)\|_{\dot{H}^{s+1}},$$

so that by virtue of (2.17), we get, by applying product laws in Besov spaces, that for any $s > -\frac{5}{2}$,

$$\|\bar{\boldsymbol{f}}(Y)(t)\|_{\dot{H}^s} \lesssim \big(1 + \|\nabla Y(t)\|_{\dot{B}^{\frac{3}{2}}}\big)^3\Big(\|\nabla Y(t)\|_{\dot{B}^{\frac{3}{2}}}\|\nabla Y_t(t)\|_{\dot{H}^{s+1}} + \|\nabla Y(t)\|_{\dot{H}^{s+1}}\|\nabla Y_t(t)\|_{\dot{B}^{\frac{3}{2}}}\Big),$$

which along with (4.11) implies that for any $s > -\frac{5}{2}$,

$$\|\bar{\boldsymbol{f}}(Y)\|_{L^2_T(\dot{H}^s)} \lesssim \|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}\|\nabla Y_t\|_{L^2_T(\dot{H}^{s+1})} + \|\nabla Y\|_{L^\infty_T(\dot{H}^{s+1})}\|\nabla Y_t\|_{L^2_T(\dot{B}^{\frac{3}{2}})}. \tag{5.9}$$

On the other hand, we get by using Bony's decomposition (3.8) that for $-\frac{5}{2} < s \leq \frac{1}{2}$,

$$\begin{aligned}\|(\mathcal{A}_Y^T - I)\nabla Y_t\|_{L^1_T(\dot{H}^{s+1})} &\lesssim \|\mathcal{A}_Y^T - I\|_{L^\infty_T(\dot{H}^{s+1})}\|\nabla Y_t\|_{L^1_T(\dot{B}^{\frac{3}{2}})}\\ &\lesssim \big(1 + \|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}\big)\|\nabla Y\|_{L^\infty_T(\dot{H}^{s+1})}\|Y_t\|_{L^1_T(\dot{B}^{\frac{5}{2}})}.\end{aligned}$$

Together with (4.11), this gives

$$\|(\mathcal{A}_Y^T - I)\nabla Y_t\|_{L^1_T(\dot{H}^{s+1})} \lesssim \|\nabla Y\|_{L^\infty_T(\dot{H}^{s+1})}\|Y_t\|_{L^1_T(\dot{B}^{\frac{5}{2}})}.$$

The same estimate holds for term $\|[(\mathcal{A}_Y - I)\mathcal{A}_Y^T\nabla Y_t](t)\|_{\dot{H}^{s+1}}$. We thus obtain for $-\frac{5}{2} < s \leq \frac{1}{2}$,

$$\|\bar{\boldsymbol{f}}(Y)\|_{L^1_T(\dot{H}^s)} \lesssim \|\nabla Y\|_{L^\infty_T(\dot{H}^{s+1})}\|Y_t\|_{L^1_T(\dot{B}^{\frac{5}{2}})}. \tag{5.10}$$

By summing up (5.8) and (5.9), we obtain for $s > -\frac{1}{2}$,

$$\|\boldsymbol{f}(Y,q)\|_{L^2_T(\dot{H}^s)} \lesssim \|\nabla q\|_{L^2_T(\dot{H}^s)} + \|Y\|_{L^\infty_T(\dot{H}^{s+2})}\big(\|\nabla q\|_{L^2_T(\dot{B}^{\frac{1}{2}})} + \|Y_t\|_{L^2_T(\dot{B}^{\frac{5}{2}})}\big) + \|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})}\|Y_t\|_{L^2_T(\dot{H}^{s+2})},$$

which together with (5.4) yields (5.6).

On the other hand, by summing up (5.8) and (5.10), we get for $-\frac{1}{2} < s \leq \frac{1}{2}$

$$\|\boldsymbol{f}(Y,q)\|_{L^1_T(\dot{H}^s)} \lesssim \|\nabla q\|_{L^1_T(\dot{H}^s)} + \|Y\|_{L^\infty_T(\dot{H}^{s+2})}\big(\|\nabla q\|_{L^1_T(\dot{B}^{\frac{1}{2}})} + \|Y_t\|_{L^1_T(\dot{B}^{\frac{5}{2}})}\big),$$

from which and (5.4), we achieve (5.7). This concludes the proof of Lemma 5.3. $\square$

5.2. **The proof of Theorem 2.2.** The proof of Theorem 2.2 is based on the following proposition:

Proposition 5.1. *Let $s_1 > \frac{5}{4}$ and $s_2 \in (-\frac{1}{2}, -\frac{1}{4})$. Let (Y, q) be a smooth enough solution of (2.21)-(2.22) on $[0, T]$. We denote*

$$\begin{aligned}
\mathcal{E}_T^{s_1,s_2}(Y,q) &\stackrel{\text{def}}{=} E_T^{s_1}(Y,q) + E_T^{s_2}(Y,q) + \|Y_t\|^2_{\widetilde{L}^\infty_T(\mathcal{B}^{\frac{1}{2},0}\cap\mathcal{B}^{s_1,0})} + \|\partial_3 Y\|^2_{\widetilde{L}^\infty_T(\mathcal{B}^{\frac{1}{2},0}\cap\mathcal{B}^{s_1,0})} \\
&\quad + \|Y\|^2_{\widetilde{L}^\infty_T(\mathcal{B}^{\frac{5}{2},0}\cap\mathcal{B}^{s_1+2,0})} + \|Y_t\|^2_{L^1_T(\mathcal{B}^{\frac{5}{2},0}\cap\mathcal{B}^{s_1+2,0})} + \|\partial_3 Y\|^2_{\widetilde{L}^2_T(\mathcal{B}^{\frac{3}{2},0}\cap\mathcal{B}^{s_1+1,0})}, \qquad (5.11)\\
\mathcal{E}_0^{s_1,s_2} &\stackrel{\text{def}}{=} E_0^{s_1} + E_0^{s_2} + \|Y_1\|^2_{\mathcal{B}^{\frac{1}{2},0}\cap\mathcal{B}^{s_1,0}} + \|\partial_3 Y_0\|^2_{\mathcal{B}^{\frac{1}{2},0}\cap\mathcal{B}^{s_1,0}} + \|Y_0\|^2_{\mathcal{B}^{\frac{5}{2},0}\cap\mathcal{B}^{s_1+2,0}},
\end{aligned}$$

where

$$\begin{aligned}
E_T^s(Y,q) \stackrel{\text{def}}{=} &\|Y_t\|^2_{\widetilde{L}^\infty_T(\dot H^s\cap\dot H^{s+1})} + \|\partial_3 Y\|^2_{\widetilde{L}^\infty_T(\dot H^s)} + \|Y\|^2_{\widetilde{L}^\infty_T(\dot H^{s+2})} \\
&+ \|Y_t\|^2_{L^2_T(\dot H^{s+1}\cap\dot H^{s+2})} + \|\partial_3 Y\|^2_{L^2_T(\dot H^{s+1})} + \|\nabla q\|^2_{L^2_T(\dot H^s)} + \|\nabla q\|^2_{L^1_T(\dot H^s)},
\end{aligned}$$

and

$$E_0^s \stackrel{\text{def}}{=} \|Y_1\|^2_{\dot H^s\cap\dot H^{s+1}} + \|\partial_3 Y_0\|^2_{\dot H^s} + \|Y_0\|^2_{\dot H^{s+2}}.$$

We assume that

$$\|\nabla Y\|_{L^\infty_T(\dot B^{\frac{3}{2}})} + \|Y\|_{L^\infty_T(\mathcal{B}^{s_1+2,0})} \le c_0 \quad \text{and} \quad \|Y\|_{L^\infty_T(\dot B^{s_1+\frac{3}{2}})} \le 1, \qquad (5.12)$$

for some c_0 sufficiently small, then there holds

$$\mathcal{E}_T^{s_1,s_2}(Y,q) \le C_1\mathcal{E}_0^{s_1,s_2} + C_1\big(\mathcal{E}_T^{s_1,s_2}(Y,q)^{1/2} + \mathcal{E}_T^{s_1,s_2}(Y,q) + \mathcal{E}_T^{s_1,s_2}(Y,q)^2\big)\mathcal{E}_T^{s_1,s_2}(Y,q), \qquad (5.13)$$

for some uniform positive constant C_1.

Proof. Under the assumptions (5.12), for $s = s_1$ and $s = s_2$, we deduce from Lemmas 5.1 and 5.2 that

$$\begin{aligned}
E_T^s(Y,q) \lesssim &E_0^s + \|Y\|^2_{L^\infty_T(\dot H^{s+2})}\big(\|\nabla q\|^2_{L^1_T(\dot B^{\frac{1}{2}})} + \|\nabla q\|^2_{L^2_T(\dot B^{\frac{1}{2}})}\big) \\
&+ \Big(\|Y_t\|^2_{L^\infty_T(\dot B^{\frac{3}{2}})} + \|Y_t\|^2_{L^2_T(\dot B^{\frac{3}{2}})} + \|\partial_3 Y\|^2_{L^\infty_T(\dot B^{\frac{3}{2}})} + \|\partial_3 Y\|^2_{L^2_T(\dot B^{\frac{3}{2}})}\Big) \\
&\times \Big(1 + \|Y_t\|^2_{L^2_T(\dot B^{\frac{3}{2}})} + \|\partial_3 Y\|^2_{L^2_T(\dot B^{\frac{3}{2}})}\Big) E_T^s(Y,q) \qquad (5.14)\\
&+ \|\boldsymbol{f}\|_{L^2_T(\dot H^s)}\|\Delta Y_t\|_{L^2_T(\dot H^s)} + \|\boldsymbol{f}\|_{\widetilde{L}^1_T(\dot H^s)}\big(\|Y_t\|_{\widetilde{L}^\infty_T(\dot H^s)} + \|\Delta Y\|_{\widetilde{L}^\infty_T(\dot H^s)}\big).
\end{aligned}$$

However, taking $s = s_1$ in (5.6) gives rise to

$$\begin{aligned}
\|\boldsymbol{f}\|_{L^2_T(\dot H^{s_1})} \lesssim &\Big(\|\nabla q\|_{L^2_T(\dot B^{\frac{1}{2}})} + \|Y_t\|_{L^\infty_T(\dot B^{\frac{3}{2}})}\big(1 + \|Y_t\|_{L^2_T(\dot B^{\frac{3}{2}})}\big) + \|\nabla Y\|_{L^\infty_T(\dot B^{\frac{3}{2}})} \\
&+ \|\partial_3 Y\|_{L^\infty_T(\dot B^{\frac{3}{2}})}\big(1 + \|\partial_3 Y\|_{L^2_T(\dot B^{\frac{3}{2}})}\big) + \|Y_t\|_{L^2_T(\dot B^{\frac{5}{2}})}\Big)\big(E_T^{s_1}(Y,q)\big)^{\frac{1}{2}},
\end{aligned}$$

and applying Proposition 4.2 and Proposition 4.3 leads to

$$\begin{aligned}
\|\boldsymbol{f}\|_{\widetilde{L}^1_T(\dot H^{s_1})} \lesssim &\|\boldsymbol{f}\|_{L^1_T(\dot H^{s_1})} \lesssim \|\boldsymbol{f}\|_{L^1_T(\mathcal{B}^{s_1,0})} \\
\lesssim &\|Y_t\|^2_{L^2_T(\dot B^{\frac{3}{4}})} + \|\partial_3 Y\|^2_{L^2_T(\dot B^{\frac{3}{4}})} + \|Y_t\|^2_{L^2_T(\dot B^{\frac{5}{2}})} + \|Y_t\|^2_{L^2_T(\dot B^{s_1+\frac{5}{4}})} + \|\partial_3 Y\|^2_{L^2_T(\dot B^{\frac{3}{2}})} \\
&+ \|\partial_3 Y\|^2_{L^2_T(\dot B^{s_1+\frac{1}{4}})} + \|Y\|^2_{L^\infty_T(\dot B^{\frac{5}{2}}\cap\mathcal{B}^{s_1+2,0})} + \|Y_t\|^2_{L^1_T(\dot B^{\frac{5}{2}}\cap\mathcal{B}^{s_1+2,0})},
\end{aligned}$$

where we used the fact that $\|Y_t\|_{L^1_T(\dot{B}^{\frac{5}{2}})} \lesssim \|Y_t\|_{L^1_T(\mathcal{B}^{\frac{5}{2},0})}$. While as $s_1 > \frac{5}{4}$ and $s_2 \in (-\frac{1}{2}, -\frac{1}{4})$, one has

$$\begin{aligned}\|Y_t\|_{L^2_T(\dot{B}^{\frac{3}{4}})} + \|\partial_3 Y\|_{L^2_T(\dot{B}^{\frac{3}{4}})} + \|Y_t\|_{L^2_T(\dot{B}^{\frac{5}{2}})} + \|Y_t\|_{L^2_T(\dot{B}^{s_1+\frac{5}{4}})} + \|\partial_3 Y\|_{L^2_T(\dot{B}^{\frac{3}{2}})} \\ + \|\partial_3 Y\|_{L^2_T(\dot{B}^{s_1+\frac{1}{4}})} \lesssim \|\partial_3 Y\|_{L^2_T(\dot{H}^{s_2+1}\cap \dot{H}^{s_1+1})} + \|Y_t\|_{L^2_T(\dot{H}^{s_2+1}\cap \dot{H}^{s_1+2})}.\end{aligned}$$

Along the same line, we deduce from Corollary 4.1 and its proof that

$$\begin{aligned}\|\nabla q\|_{L^1_T(\dot{B}^{\frac{1}{2}})} + \|\nabla q\|_{L^2_T(\dot{B}^{\frac{1}{2}})} \lesssim & \|Y_t\|_{L^2_T(\dot{B}^{\frac{3}{2}})}\big(\|Y_t\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})} + \|Y_t\|_{L^2_T(\dot{B}^{\frac{3}{2}})}\big) \\ & + \|\partial_3 Y\|_{L^2_T(\dot{B}^{\frac{3}{2}})}\big(\|\partial_3 Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})} + \|\partial_3 Y\|_{L^2_T(\dot{B}^{\frac{3}{2}})}\big) \lesssim \mathcal{E}^{s_1,s_2}_T(Y,q).\end{aligned}$$

As a consequence, we obtain

$$\|\boldsymbol{f}\|_{L^2_T(\dot{H}^{s_1})} + \|\boldsymbol{f}\|_{\widetilde{L}^1_T(\dot{H}^{s_1})} \lesssim \Big(\mathcal{E}^{s_1,s_2}_T(Y,q) + \mathcal{E}^{s_1,s_2}_T(Y,q)^{\frac{1}{2}}\Big) E^{s_1}_T(Y,q)^{\frac{1}{2}} + \mathcal{E}^{s_1,s_2}_T(Y,q).$$

Resuming the above estimates into (5.14) yields

$$\begin{aligned}E^{s_1}_T(Y,q) \lesssim & E^{s_1}_0 + \big(\mathcal{E}^{s_1,s_2}_T(Y,q)^{\frac{1}{2}} + \mathcal{E}^{s_1,s_2}_T(Y,q) + \mathcal{E}^{s_1,s_2}_T(Y,q)^2\big) E^{s_1}_T(Y,q) \\ & + \mathcal{E}^{s_1,s_2}_T(Y,q) E^{s_1}_T(Y,q)^{\frac{1}{2}}.\end{aligned} \tag{5.15}$$

On the other hand, it follows from Lemma 5.3 and the fact that $\|Y_t\|_{L^1_T(\dot{B}^{\frac{5}{2}})} \lesssim \|Y_t\|_{L^1_T(\mathcal{B}^{\frac{5}{2},0})}$, that

$$\|\boldsymbol{f}\|_{L^2_T(\dot{H}^{s_2})} + \|\boldsymbol{f}\|_{\widetilde{L}^1_T(\dot{H}^{s_2})} \lesssim \big(\mathcal{E}^{s_1,s_2}_T(Y,q)^{\frac{1}{2}} + \mathcal{E}^{s_1,s_2}_T(Y,q)\big) E^{s_1}_T(Y,q)^{\frac{1}{2}}$$

from which, we infer, by a similar derivation of (5.15), that

$$E^{s_2}_T(Y,q) \lesssim E^{s_2}_0 + \big(\mathcal{E}^{s_1,s_2}_T(Y,q)^{\frac{1}{2}} + \mathcal{E}^{s_1,s_2}_T(Y,q) + \mathcal{E}^{s_1,s_2}_T(Y,q)^2\big) E^{s_2}_T(Y,q). \tag{5.16}$$

Therefore, by summing up (5.15), (5.16) and (4.33) with $s = s_1$, we achieve (5.13). This completes the proof of Proposition 5.1. □

Now we are in a position to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Given initial data (Y_0, Y_1) satisfying the assumptions listed in Theorem 2.2, we deduce by Proposition 5.1 and a standard argument that (2.21)-(2.22) has a unique solution (Y, q) on $[0, T]$, which satisfies (2.26) on $[0, T]$. Let T^* be the largest possible time so that (2.26) holds. Then to complete the proof of Theorem 2.2, we only need to show that $T^* = \infty$ and there holds (2.27) under the assumptions of (2.24) and (2.25). Otherwise, if $T^* < \infty$, we denote

$$\bar{T} \stackrel{\text{def}}{=} \max\big\{\ T < T^* :\ \ \mathcal{E}^{s_1,s_2}_T(Y,q) \leq \eta_0^2\ \ \big\}, \tag{5.17}$$

for η_0 so small that $C_1(\eta_0 + \eta_0^2 + \eta_0^4) \leq \frac{1}{2}$, and

$$\|\nabla Y\|_{L^\infty_T(\dot{B}^{\frac{3}{2}})} + \|Y\|_{L^\infty_T(\mathcal{B}^{s_1+2,0})} + \|Y\|_{L^\infty_T(\dot{B}^{s_1+\frac{3}{2}})} \leq C_2 \mathcal{E}^{s_1,s_2}_T(Y,q)^{\frac{1}{2}} \leq C_2\eta_0 \leq 1, \tag{5.18}$$

for the same C_1 as that in (5.13).

We shall prove that $\bar{T} = \infty$ provided that ε_0 is sufficiently small in (2.25). In fact, thanks to (5.18), we get by applying Proposition 5.1 that

$$\mathcal{E}^{s_1,s_2}_{\bar{T}}(Y,q) \leq 2C_1 \mathcal{E}^{s_1,s_2}_0. \tag{5.19}$$

In particular, if we take ε_0 so small that $2C_1\varepsilon_0^2 \leq \frac{1}{2}\eta_0^2$, (5.19) contradicts with (5.17) if $\bar{T} < \infty$. This in turn shows that $\bar{T} = T^* = \infty$, and there holds (2.27). This completes the proof of Theorem 2.2. $\square$

6. The proof of Theorem 2.1 and Theorem 1.1

With Theorem 2.2 and Lemma A.1 in the Appendix A in hand, we can now present the proof of Theorem 2.1.

Proof of Theorem 2.1. Under the assumptions of Theorem 2.1, we get, by applying Lemma 2.1, that there exists a vector-valued function $Y_0(y) = (Y_0^1(y), Y_0^2(y), Y_0^3(y))^T$ so that

$$X_0(y) = I + Y_0(y) \quad \text{and} \quad U_0 \circ X_0(y) = \nabla_y X_0(y) = I + \nabla_y Y_0(y),$$

which in particular implies

$$\frac{\partial X_0^{-1}(x)}{\partial x} = (I + \nabla_y Y_0)^{-1} \circ X_0^{-1}(x) = U_0^{-1}(x) = I - \nabla_x \mathbf{\Psi}. \tag{6.1}$$

Let $Y_1(y) \stackrel{\text{def}}{=} \boldsymbol{u}_0(X_0(y))$. Applying Lemma A.1 with $\Phi(x) = X_0^{-1}(x)$ gives

$$\begin{aligned}\|Y_1\|_{\dot{H}^{s_1+1}} + \|Y_1\|_{\dot{H}^{s_2}} \leq & C\big(\|\nabla\mathbf{\Psi}\|_{\dot{B}^{\frac{3}{2}}}\big)\Big(\|\boldsymbol{u}_0\|_{\dot{H}^{s_2}} + \|\boldsymbol{u}_0\|_{\dot{H}^{s_1+1}} + \|\nabla\mathbf{\Psi}\|_{\dot{H}^{s_1+\frac{1}{2}}}\|\boldsymbol{u}_0\|_{\dot{H}^2} \\ & + (1 + \|\Delta\mathbf{\Psi}\|_{H^{s_1-1}})\|\nabla\boldsymbol{u}_0\|_{H^{s_1}}\Big) \\ \leq & C\big(\|\nabla\mathbf{\Psi}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\Big(\|\boldsymbol{u}_0\|_{\dot{H}^{s_2}} + \big(1 + \|\nabla\mathbf{\Psi}\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,2}}\big)\|\nabla\boldsymbol{u}_0\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,2}}\Big),\end{aligned}$$

where in the last step, we used Lemma 3.1 so that

$$\|\nabla\boldsymbol{u}_0\|_{H^{s_1}} \lesssim \|\nabla\boldsymbol{u}_0\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}} \quad \text{for} \quad p \in (1,2).$$

Similarly as $\|\partial_3 Y_0\|_{\mathcal{B}^{\frac{1}{2},0}} \lesssim \|Y_0\|_{\mathcal{B}^{\frac{1}{2},1}} \lesssim \|Y_0\|_{\dot{B}^{\frac{3}{2}}}$, and by virtue of (2.12), $Y_0 = \mathbf{\Psi}(X_0(y))$, one has

$$\begin{aligned}&\|Y_0\|_{\dot{H}^{s_2+2}\cap\dot{H}^{s_1+2}} + \|\partial_3 Y_0\|_{\dot{H}^{s_2}} + \|\partial_3 Y_0\|_{\mathcal{B}^{\frac{1}{2},0}} \\ &\qquad \leq C\big(\|\nabla\mathbf{\Psi}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\Big(\|\mathbf{\Psi}\|_{\dot{B}^{s_2+\frac{3}{p}-\frac{1}{2}}_{p,2}} + \big(1 + \|\Delta\mathbf{\Psi}\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,2}}\big)\|\nabla\mathbf{\Psi}\|_{B^{s_1+\frac{3}{p}-\frac{1}{2}}_{p,2}}\Big),\end{aligned}$$

Whereas as $p < 2$, it follows from Definition 3.2 and Lemma 3.1 that

$$\begin{aligned}\|a\|_{\mathcal{B}^{s,0}} = \sum_{j,k\in\mathbb{Z}^2} 2^{js}\|\Delta_j\Delta_k^v a\|_{L^2} &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(s+\frac{2}{p}-1)}\|\Delta_j a\|_{L^p}\Big(\sum_{k\leq j+N_0} 2^{k(\frac{1}{p}-\frac{1}{2})}\Big) \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(s+\frac{3}{p}-\frac{3}{2})}\|\Delta_j a\|_{L^p} \lesssim \|a\|_{\dot{B}^{s+\frac{3}{p}-\frac{3}{2}}_{p,1}}.\end{aligned} \tag{6.2}$$

Thanks to (6.2), we get, by applying Lemma A.1, that

$$\begin{aligned}\|Y_0\|_{\mathcal{B}^{\frac{5}{2},0}\cap\mathcal{B}^{s_1+2,0}} \lesssim & \|Y_0\|_{\dot{B}^{1+\frac{3}{p}}_{p,1}\cap\dot{B}^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1}} \\ \leq & C\big(\|\nabla\mathbf{\Psi}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\Big(\|\mathbf{\Psi}\|_{\dot{B}^{1+\frac{3}{p}}_{p,1}} + \big(1 + \|\Delta\mathbf{\Psi}\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}}\big)\|\nabla\mathbf{\Psi}\|_{B^{s_1+\frac{3}{p}-\frac{1}{2}}_{p,1}}\Big),\end{aligned}$$

and

$$\begin{aligned}\|Y_1\|_{\mathcal{B}^{\frac12,0}\cap\mathcal{B}^{s_1,0}} \lesssim&\|Y_1\|_{\dot B^{\frac3p-1}_{p,1}\cap\dot B^{s_1+\frac3p-\frac32}_{p,1}}\\ \leq& C\big(\|\nabla\boldsymbol{\Psi}\|_{\dot B^{\frac3p}_{p,1}}\big)\Big(\|\boldsymbol{u}_0\|_{\dot B^{\frac3p-1}_{p,1}}+\big(1+\|\nabla\boldsymbol{\Psi}\|_{B^{s_1+\frac3p-\frac52}_{p,1}}\big)\|\nabla\boldsymbol{u}_0\|_{B^{s_1+\frac3p-\frac52}_{p,1}}\Big).\end{aligned}$$

Therefore, thanks to (2.7), we conclude that

$$\begin{aligned}&\|Y_0\|_{\dot H^{s_1+2}\cap\dot H^{s_2+2}}+\|Y_0\|_{\mathcal{B}^{s_1+2,0}\cap\mathcal{B}^{\frac52,0}}+\|\partial_3Y_0\|_{\dot H^{s_2}}+\|\partial_3Y_0\|_{\mathcal{B}^{s_1,0}\cap\mathcal{B}^{\frac12,0}}\\ &\qquad+\|Y_1\|_{\dot H^{s_1+1}}+\|Y_1\|_{\dot H^{s_2}}+\|Y_1\|_{\mathcal{B}^{s_1,0}\cap\mathcal{B}^{\frac12,0}}\\ &\leq C\big(\|\nabla\boldsymbol{\Psi}\|_{\dot B^{s_2+\frac3p-\frac32}_{p,2}\cap B^{s_1+\frac3p-\frac12}_{p,1}}+\|\boldsymbol{u}_0\|_{\dot H^{s_2}\cap\dot B^{\frac3p-1}_{p,1}}+\|\nabla\boldsymbol{u}_0\|_{B^{s_1+\frac3p-\frac32}_{p,1}}\big)\leq C\varepsilon_0,\end{aligned}\tag{6.3}$$

from which, and Theorem 2.2, we deduce that the system (2.20) (equivalently (2.21)-(2.22)) has a unique global solution (Y,q) which satisfies (2.26) and (2.27) provided that ε_0 in (2.7) is sufficiently small.

We denote $X(t,y)\overset{\text{def}}{=}y+Y(t,y)$. Then it follows from (2.27) that $X(t,y)$ is invertible with respect to y variables and we denote its inverse mapping by $X^{-1}(t,x)$. Since $\det(1+\nabla Y)=1$, the adjoint matrix $\mathcal{A}_Y$ of $I+\nabla Y$ satisfies

$$\nabla\cdot\mathcal{A}_Y=\mathbf{0}\quad\text{and}\quad\mathcal{A}_Y=(I+\nabla Y)^{-1}$$

which implies

$$\nabla_x\cdot[(I+\nabla Y)\circ X^{-1}]=\big(\nabla\cdot[\mathcal{A}_Y(I+\nabla Y)]\big)\circ X^{-1}=\mathbf{0}.\tag{6.4}$$

Then we define $U(t,x)=\big(\bar{\boldsymbol{b}}(t,x),\tilde{\boldsymbol{b}}(t,x),\boldsymbol{b}(t,x)\big)$ and $(\boldsymbol{u}(t,x),p(t,x))$ through

$$\begin{aligned}&U(t,x)=(\bar{\boldsymbol{b}},\tilde{\boldsymbol{b}},\boldsymbol{b})(t,x)\overset{\text{def}}{=}(I+\nabla Y)(t,X^{-1}(t,x))\quad\text{and}\\ &\boldsymbol{u}(t,x)\overset{\text{def}}{=}Y_t(t,X^{-1}(t,x)),\quad p(t,x)\overset{\text{def}}{=}q(t,X^{-1}(t,x))-\frac12|\boldsymbol{b}(t,x)|^2,\end{aligned}\tag{6.5}$$

from which and (6.4), we infer that

$$\mathrm{div}\bar{\boldsymbol{b}}=\mathrm{div}\tilde{\boldsymbol{b}}=\mathrm{div}\boldsymbol{b}=0.$$

Hence according to Section 2, $(U,\boldsymbol{u},p)$ thus defined globally solves (2.6). Then to complete the proof of Theorem 2.1, it amounts to prove (2.9). For this, we first notice from (6.5) that

$$(\nabla\boldsymbol{u})\circ X(t,y)=\nabla_yY_t(t,y)\big(I+\nabla_yY(t,y)\big)^{-1},$$

which along with the proof of (A.4) in the Appendix A implies

$$\begin{aligned}\|\boldsymbol{u}\|_{L^1(\mathbb{R}^+;\dot B^{\frac52})}&\lesssim(1+\|\nabla_yY\|_{L^\infty(\mathbb{R}^+;\dot B^{\frac32})})^2\|\nabla_yY_t\|_{L^1(\mathbb{R}^+;\dot B^{\frac32})}\\ &\leq C(\|\nabla Y\|_{L^\infty(\mathbb{R}^+;\dot B^{\frac32})})\|Y_t\|_{L^1(\mathbb{R}^+;\mathcal{B}^{\frac52,0})}.\end{aligned}\tag{6.6}$$

Again thanks to (6.5), we get, by applying Lemma A.1 with $\Phi=X(t,y)$, that

$$\begin{aligned}&\|(\bar{\boldsymbol{b}}-\boldsymbol{e}_1,\tilde{\boldsymbol{b}}-\boldsymbol{e}_2)\|_{L^\infty(\mathbb{R}^+;\dot H^{s_2+1})}+\|\boldsymbol{b}-\boldsymbol{e}_3\|_{L^\infty(\mathbb{R}^+;\dot H^{s_2})}\\ &\qquad+\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^+;\dot H^{s_2})}+\|\boldsymbol{b}-\boldsymbol{e}_3\|_{L^2(\mathbb{R}^+;\dot H^{s_2+1})}+\|\boldsymbol{u}\|_{L^2(\mathbb{R}^+;\dot H^{s_2+1})}\\ &\leq C(\|\nabla Y\|_{L^\infty(\mathbb{R}^+;\dot B^{\frac32})})\Big(\|(\partial_1Y,\partial_2Y)\|_{L^\infty(\mathbb{R}^+;\dot H^{s_2+1})}+\|\partial_3Y\|_{L^\infty(\mathbb{R}^+;\dot H^{s_2})}\\ &\qquad+\|Y_t\|_{L^\infty(\mathbb{R}^+;\dot H^{s_2})}+\|\partial_3Y\|_{L^2(\mathbb{R}^+;\dot H^{s_2+1})}+\|Y_t\|_{L^2(\mathbb{R}^+;\dot H^{s_2+1})}\Big).\end{aligned}\tag{6.7}$$

Along the same line, one has

$$\|\boldsymbol{u}\|_{L^1(\mathbb{R}^+;\dot{H}^{s_1+2})} \leq C(\|\nabla Y\|_{L^\infty(\mathbb{R}^+;\dot{B}^{\frac{3}{2}})})(1+\|\Delta Y\|_{L^\infty(\mathbb{R}^+;H^{s_1})})\|\nabla Y_t\|_{L^1(\mathbb{R}^+;H^{s_1+1})}, \tag{6.8}$$

and

$$\begin{aligned}
&\|(\bar{\boldsymbol{b}}-\boldsymbol{e}_1,\tilde{\boldsymbol{b}}-\boldsymbol{e}_2)\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1+1})}+\|\boldsymbol{b}-\boldsymbol{e}_3\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1+1})}\\
&\quad+\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1+1})}+\|\boldsymbol{b}-\boldsymbol{e}_3\|_{L^2(\mathbb{R}^+;\dot{H}^{s_1+1})}+\|\boldsymbol{u}\|_{L^2(\mathbb{R}^+;\dot{H}^{s_1+2})}\\
&\leq C(\|\nabla Y\|_{L^\infty(\mathbb{R}^+;\dot{B}^{\frac{3}{2}})})\Big(\|\nabla Y\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1+1})}+\|Y_t\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1+1})}\\
&\quad+\|\partial_3 Y\|_{L^2(\mathbb{R}^+;\dot{H}^{s_1+1})}+\|Y\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1+\frac{3}{2}})}\big(\|\nabla Y\|_{L^\infty(\mathbb{R}^+;\dot{H}^2)}+\|Y_t\|_{L^\infty(\mathbb{R}^+;\dot{H}^2)}\\
&\quad+\|\partial_3 Y\|_{L^2(\mathbb{R}^+;\dot{H}^2)}\big)+(1+\|\Delta Y\|_{L^\infty(\mathbb{R}^+;H^{s_1})})\big(\|\Delta Y\|_{L^\infty(\mathbb{R}^+;H^{s_1})}\\
&\quad+\|\nabla Y_t\|_{L^\infty(\mathbb{R}^+;H^{s_1})}+\|\partial_3\nabla Y\|_{L^2(\mathbb{R}^+;H^{s_1})}+\|\nabla Y_t\|_{L^2(\mathbb{R}^+;H^{s_1+1})}\big)\Big).
\end{aligned} \tag{6.9}$$

Consequently, we deduce from (2.27), (3.6), (6.3), (6.6) to (6.9) and the fact that $\|u\|_{\dot{H}^s} \lesssim \|u\|_{\mathcal{B}^{s,0}}$ that

$$\begin{aligned}
&\|(\bar{\boldsymbol{b}}-\boldsymbol{e}_1,\tilde{\boldsymbol{b}}-\boldsymbol{e}_2)\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1+1}\cap\dot{H}^{s_2+1})}+\|\boldsymbol{b}-\boldsymbol{e}_3\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1+1}\cap\dot{H}^{s_2})}\\
&\quad+\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1+1}\cap\dot{H}^{s_2})}+\|\boldsymbol{b}-\boldsymbol{e}_3\|_{L^2(\mathbb{R}^+;\dot{H}^{s_1+1}\cap\dot{H}^{s_2+1})}\\
&\quad+\|\boldsymbol{u}\|_{L^2(\mathbb{R}^+;\dot{H}^{s_1+2}\cap\dot{H}^{s_2+1})}+\|\boldsymbol{u}\|_{L^1(\mathbb{R}^+;\dot{H}^{s_1+2}\cap\dot{B}^{\frac{5}{2}})}\\
&\leq C\big(\|\nabla\boldsymbol{\Psi}\|_{\dot{B}^{s_2+\frac{3}{p}-\frac{3}{2}}_{p,2}\cap B^{s_1+\frac{3}{p}-\frac{1}{2}}_{p,1}}+\|\boldsymbol{u}_0\|_{\dot{H}^{s_2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1}}+\|\nabla\boldsymbol{u}_0\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}}\big),
\end{aligned} \tag{6.10}$$

provided that ε_0 is sufficiently small in (2.7).

On the other hand, taking space divergence to the momentum equations of (1.1) gives rise to

$$\nabla p = -\frac{1}{2}\nabla(|\boldsymbol{b}|^2)+\nabla(-\Delta)^{-1}\mathrm{div}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\boldsymbol{b}\cdot\nabla\boldsymbol{b}), \tag{6.11}$$

then applying product laws in Sobolev spaces gives rise to

$$\begin{aligned}
&\|\nabla p\|_{L^2(\mathbb{R}^+;\dot{H}^{s_1}\cap\dot{H}^{s_2})}\\
&\leq C\big(\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^+;\dot{B}^{\frac{3}{2}})}\|\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}^+;\dot{H}^{s_1}\cap\dot{H}^{s_2})}+\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1}\cap\dot{H}^{s_2})}\|\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}^+;\dot{B}^{\frac{3}{2}})}\\
&\quad+(1+\|\boldsymbol{b}-\boldsymbol{e}_3\|_{L^\infty(\mathbb{R}^+;\dot{B}^{\frac{3}{2}})})\|\nabla\boldsymbol{b}\|_{L^2(\mathbb{R}^+;\dot{H}^{s_1}\cap\dot{H}^{s_2})}+\|\boldsymbol{b}-\boldsymbol{e}_3\|_{L^\infty(\mathbb{R}^+;\dot{H}^{s_1+1})}\|\nabla\boldsymbol{b}\|_{L^2(\mathbb{R}^+;\dot{B}^{\frac{1}{2}})}\big),
\end{aligned}$$

which together with (6.10) ensures that

$$\|\nabla p\|_{L^2(\mathbb{R}^+;\dot{H}^{s_1}\cap\dot{H}^{s_2})} \leq C\big(\|\nabla\boldsymbol{\Psi}\|_{\dot{B}^{s_2+\frac{3}{p}-\frac{3}{2}}_{p,2}\cap B^{s_1+\frac{3}{p}-\frac{1}{2}}_{p,1}}+\|\boldsymbol{u}_0\|_{\dot{H}^{s_2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1}}+\|\nabla\boldsymbol{u}_0\|_{B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}}\big),$$

provided that ε_0 is sufficiently small in (2.7). This completes the proof of (2.9) and thus Theorem 2.1. $\square$

Before we present the proof of Theorem 1.1, we shall first prove the following blow-up criterion for smooth enough solutions of (1.1).

Proposition 6.1. *Let $\boldsymbol{b}_0-\boldsymbol{e}_3 \in H^s(\mathbb{R}^3)$ and $\boldsymbol{u}_0 \in H^s(\mathbb{R}^3)$ for $s>\frac{3}{2}$, (1.1) has a unique solution $(\boldsymbol{b},\boldsymbol{u})$ on $[0,T]$ for some $T>0$ so that $\boldsymbol{b}-\boldsymbol{e}_3 \in C([0,T];H^s(\mathbb{R}^3))$, $\boldsymbol{u} \in C([0,T];H^s(\mathbb{R}^3))$*

with $\nabla \boldsymbol{u} \in L^2((0,T);\dot{H}^{s+1}(\mathbb{R}^3))$ *and* $\nabla p \in C([0,T];H^{s-1}(\mathbb{R}^3))$. *Moreover, if* T^* *is the life span to this solution, and* $T^* < \infty$, *one has*

$$\int_0^{T^*} \big(\|\nabla \boldsymbol{u}(t)\|_{L^\infty} + \|\boldsymbol{b}(t)\|_{L^\infty}^2\big)\, dt = \infty. \tag{6.12}$$

Proof. It is well-known that the existence of solution to a nonlinear PDE basically follows from the uniform estimates to some smooth enough approximate solutions. For simplicity, we may only present *a priori* estimates to smooth enough solutions of (1.1) (one may check [23] for the detailed proof to the related system). As a matter of fact, let $\vec{b} \stackrel{\text{def}}{=} \boldsymbol{b} - \boldsymbol{e}_3$, we first get, by using a standard energy estimate for (1.1), that

$$\frac{1}{2}\frac{d}{dt}\big(\|\vec{b}\|_{L^2}^2 + \|\boldsymbol{u}\|_{L^2}^2\big) + \|\nabla \boldsymbol{u}\|_{L^2}^2 = 0. \tag{6.13}$$

Along the same line, applying Δ_j to the system (1.1) and then taking L^2 inner product of the resulting equations with $(\Delta_j \vec{b}, \Delta_j \boldsymbol{u})$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Delta_j \vec{b}\|_{L^2}^2 + \|\Delta_j \boldsymbol{u}\|_{L^2}^2) + \|\nabla \Delta_j \boldsymbol{u}\|_{L^2}^2\\
&\quad = -\big(\Delta_j(\boldsymbol{u}\cdot\nabla \vec{b}) \mid \Delta_j \vec{b}\big) + \big(\Delta_j(\vec{b}\cdot\nabla \boldsymbol{u}) \mid \Delta_j \vec{b}\big)\\
&\qquad - \big(\Delta_j(\boldsymbol{u}\cdot\nabla \boldsymbol{u}) \mid \Delta_j \boldsymbol{u}\big) + \big(\Delta_j(\vec{b}\cdot\nabla \vec{b}) \mid \Delta_j \boldsymbol{u}\big).
\end{aligned} \tag{6.14}$$

By virtue of the commutator estimates (see Section 2.10 of [2]), we write

$$\begin{aligned}
&\big|\big(\Delta_j(\boldsymbol{u}\cdot\nabla \vec{b}) \mid \Delta_j \vec{b}\big)\big| \lesssim c_j(t)^2 2^{-2js}\big(\|\nabla \boldsymbol{u}(t)\|_{L^\infty}\|\vec{b}(t)\|_{\dot{H}^s}^2 + \|\vec{b}(t)\|_{L^\infty}\|\nabla \boldsymbol{u}(t)\|_{\dot{H}^s}\|\vec{b}(t)\|_{\dot{H}^s}\big),\\
&\big|\big(\Delta_j(\boldsymbol{u}\cdot\nabla \boldsymbol{u}) \mid \Delta_j \boldsymbol{u}\big)\big| \lesssim c_j(t)^2 2^{-2js}\|\nabla \boldsymbol{u}(t)\|_{L^\infty}\|\boldsymbol{u}(t)\|_{\dot{H}^s}^2 \quad \text{for any} \quad s>0.
\end{aligned}$$

Whereas it follows from product laws in Sobolev spaces that

$$\big|\big(\Delta_j(\vec{b}\cdot\nabla \boldsymbol{u}) \mid \Delta_j \vec{b}\big)\big| \lesssim c_j(t)^2 2^{-2js}\big(\|\vec{b}(t)\|_{L^\infty}\|\nabla \boldsymbol{u}(t)\|_{\dot{H}^s} + \|\nabla \boldsymbol{u}(t)\|_{L^\infty}\|\vec{b}(t)\|_{\dot{H}^s}\big)\|\vec{b}(t)\|_{\dot{H}^s},$$

and

$$\big|\big(\Delta_j(\vec{b}\cdot\nabla \vec{b}) \mid \Delta_j \boldsymbol{u}\big)\big| = \big|\big(\Delta_j(\vec{b}\otimes \vec{b}) \mid \Delta_j \nabla \boldsymbol{u}\big)\big| \lesssim c_j(t)^2 2^{-2js}\|\vec{b}(t)\|_{L^\infty}\|\vec{b}(t)\|_{\dot{H}^s}\|\nabla \boldsymbol{u}(t)\|_{\dot{H}^s}.$$

Resuming the above estimates into (6.14) and using (6.13), we conclude that for any $s>0$,

$$\begin{aligned}
\|\boldsymbol{u}(t)\|_{H^s}^2 + \|\vec{b}(t)\|_{H^s}^2 + \|\nabla \boldsymbol{u}\|_{L^2_t(H^s)}^2 &\leq \|\boldsymbol{u}_0\|_{\dot{H}^s}^2 + \|\vec{b}_0\|_{\dot{H}^s}^2\\
&\quad + C\int_0^t \big(\|\nabla \boldsymbol{u}(t')\|_{L^\infty} + \|\vec{b}(t')\|_{L^\infty}^2\big)\big(\|\boldsymbol{u}(t')\|_{\dot{H}^s}^2 + \|\vec{b}(t')\|_{\dot{H}^s}^2\big)\, dt'.
\end{aligned} \tag{6.15}$$

Notice that $s>\frac{3}{2}$, one has, $\|\nabla \boldsymbol{u}(t)\|_{L^\infty} \lesssim \|\nabla \boldsymbol{u}(t)\|_{H^s}$, we thus achieve

$$\begin{aligned}
\|\boldsymbol{u}(t)\|_{H^s}^2 + \|\vec{b}(t)\|_{H^s}^2 + \|\nabla \boldsymbol{u}\|_{L^2_t(H^s)}^2 &\leq \|\boldsymbol{u}_0\|_{\dot{H}^s}^2 + \|\vec{b}_0\|_{\dot{H}^s}^2\\
&\quad + C\int_0^t \big(\|\boldsymbol{u}(t')\|_{H^s}^2 + \|\vec{b}(t')\|_{H^s}^2\big)\big(\|\boldsymbol{u}(t')\|_{\dot{H}^s}^2 + \|\vec{b}(t')\|_{\dot{H}^s}^2\big) dt',
\end{aligned}$$

from which, we infer that there exists a positive time T^*, so that there holds

$$\|\boldsymbol{u}\|_{L^\infty_T(H^s)}^2 + \|\vec{b}(t)\|_{L^\infty_T(H^s)}^2 + \|\nabla \boldsymbol{u}\|_{L^2_T(H^s)}^2 \leq C_T(\|\boldsymbol{u}_0\|_{\dot{H}^s}^2 + \|\vec{b}_0\|_{\dot{H}^s}^2) \quad \text{for any} \quad T<T^*. \tag{6.16}$$

which along with (6.11) ensures that $\nabla p \in C([0,T];H^{s-1}(\mathbb{R}^3))$ for any $T<T^*$. This concludes the existence part of Proposition 6.1.

Finally, applying Gronwall's inequality to (6.15) yields

$$\begin{aligned}&\|\boldsymbol{u}(t)\|_{H^s}^2+\|\vec{b}(t)\|_{H^s}^2+\|\nabla\boldsymbol{u}\|_{L^2_t(H^s)}^2\\&\qquad\leq(\|\boldsymbol{u}_0\|_{H^s}^2+\|\vec{b}_0\|_{H^s}^2)\exp\Big(C\int_0^t\big(\|\nabla\boldsymbol{u}(t')\|_{L^\infty}+\|\vec{b}(t')\|_{L^\infty}^2\big)\,dt'\Big)\quad\text{for}\quad t<T^*,\end{aligned}$$

which together with a classical continuous argument implies (6.12). This completes the proof of Proposition 6.1. □

Now we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Under the assumption of Theorem 1.1, we deduce from Proposition 2.1 that there exists a $\boldsymbol{\Psi}=(\psi_1,\psi_2,\psi_3)^T$ so that there holds (2.1)-(2.5). Notice that for $s_2\in(-\frac{1}{2},-\frac{1}{4})$ and $p\in(1,2)$, $s_2+\frac{3}{p}-\frac{1}{2}>0$, then it is easy to observe that

$$\|\nabla\boldsymbol{\Psi}\|_{\dot{B}^{s_2+\frac{3}{p}-\frac{3}{2}}_{p,2}\cap B^{s_1+\frac{3}{p}-\frac{1}{2}}_{p,1}}\lesssim\|\boldsymbol{\Psi}\|_{B^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1}}\lesssim\|\boldsymbol{b}_0-\boldsymbol{e}_3\|_{B^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1}}.$$

Therefore, under the assumption of (1.16), for $U_0=(I-\nabla\boldsymbol{\Psi})^{-1}$, we infer from Theorem 2.1 that (2.6) has a unique global solution $(U,\boldsymbol{u},p)$ so that there holds (2.8) and (2.9). Let $U=(\bar{\boldsymbol{b}},\tilde{\boldsymbol{b}},\boldsymbol{b})$, then according to the discussions at the beginning of Section 2, $(\boldsymbol{b},\boldsymbol{u},p)$ thus obtained solves (1.1), which is in fact the unique solution of (1.1) with initial data $(\boldsymbol{b}_0,\boldsymbol{u}_0)$, and there holds (1.18).

On the other hand, by virtue of Proposition 6.1, given initial data $(\boldsymbol{b}_0,\boldsymbol{u}_0)$ with $\boldsymbol{b}_0-\boldsymbol{e}_3\in H^s(\mathbb{R}^3)$ and $\boldsymbol{u}_0\in H^s(\mathbb{R}^3)$ (since $\boldsymbol{u}_0\in\dot{H}^{s_2}(\mathbb{R}^3)$ and $\nabla\boldsymbol{u}_0\in H^{s-1}(\mathbb{R}^3)$) for $s\geq s_1+2$, (1.1) has a unique solution $(\boldsymbol{b},\boldsymbol{u},p)$ with $\boldsymbol{b}-\boldsymbol{e}_3\in C([0,T];H^s(\mathbb{R}^3))$, $\nabla p\in C([0,T];H^{s-1}(\mathbb{R}^3))$, and $\boldsymbol{u}\in C([0,T];H^s(\mathbb{R}^3))$ **with** $\nabla\boldsymbol{u}\in L^2((0,T);\dot{H}^{s+1}(\mathbb{R}^3))$ for any fixed $T<T^*$. Furthermore, if $T^*<\infty$, there holds (6.12). Due to the uniqueness, this solution must coincide with the one obtained in the last paragraph. Then thanks to (1.18), (6.12) can not be true for any finite T^*, and therefore $T^*=\infty$ and there holds (1.17). This completes the proof of Theorem 1.1.

□

Appendix A. The Besov estimates to functions composed with a measure preserving diffeomorphism

Lemma A.1. *Let $\Phi(y)=y+\Psi(y)$ be a smooth volume preserving diffeomorphism on $\mathbb{R}^3$. Then for $u,v\in\mathcal{S}(\mathbb{R}^3)$, there hold*

$$\begin{aligned}&\|u\circ\Phi\|_{\dot{B}^s_{p,r}}\leq C\big(\|\nabla\Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\|u\|_{\dot{B}^s_{p,r}}\quad\textit{and}\\&\|v\circ\Phi^{-1}\|_{\dot{B}^s_{p,r}}\leq C\big(\|\nabla\Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\|v\|_{\dot{B}^s_{p,r}}\qquad\textit{for}\quad s\in(-1,2],\\&\|u\circ\Phi\|_{\dot{B}^s_{p,r}}\leq C\big(\|\nabla\Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\big(\|u\|_{\dot{B}^s_{p,r}}+\|\Psi\|_{\dot{B}^{s+\frac{1}{2}}_{p,r}}\|u\|_{\dot{B}^2_{p,r}}\big)\quad\textit{and}\\&\|v\circ\Phi^{-1}\|_{\dot{B}^s_{p,r}}\leq C\big(\|\nabla\Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\big(\|v\|_{\dot{B}^s_{p,r}}+\|\Psi\|_{\dot{B}^{s+\frac{1}{2}}_{p,r}}\|v\|_{\dot{B}^2_{p,r}}\big)\qquad\textit{for}\quad s\in(2,3],\\&\|u\circ\Phi\|_{\dot{B}^s_{p,r}}\leq C\big(\|\nabla\Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\big(1+\|\Delta\Psi\|_{B^{s-2}_{p,r}}\big)\|\nabla u\|_{B^{s-1}_{p,r}}\quad\textit{and}\\&\|v\circ\Phi^{-1}\|_{\dot{B}^s_{p,r}}\leq C\big(\|\nabla\Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\big(1+\|\Delta\Psi\|_{B^{s-2}_{p,r}}\big)\|\nabla v\|_{B^{s-1}_{p,r}}\qquad\textit{for}\quad s>3,\end{aligned}\tag{A.1}$$

where $C(\lambda)$ denotes a positive constant non-decreasingly depending on λ.

Proof. Let
$$\mathcal{A}=(a_{ij})_{i,j=1,2,3}\stackrel{\text{def}}{=}I+\nabla_y\Psi,\quad \mathcal{B}=(b_{ij})_{i,j=1,2,3}\stackrel{\text{def}}{=}(I+\nabla_y\Psi)^{-1}.$$
Then due to $\det\mathcal{A}=1$, the matrix $\mathcal{B}$ equals the adjoint matrix of $\mathcal{A}$. This leads to
$$(\partial_{x_i}u)\circ\Phi=\sum_{j=1}^3 b_{ji}\partial_{y_j}(u\circ\Phi)\quad\text{and}\quad(\partial_{y_i}v)\circ\Phi^{-1}=\sum_{j=1}^3 a_{ji}\circ\Phi^{-1}\partial_{x_j}(v\circ\Phi^{-1}).\tag{A.2}$$

In what follows, we shall only present the proof of the related estimates involving $u\circ\Phi$, and the ones involving $v\circ\Phi^{-1}$ are identical. We first deduce from Lemma 2.7 of [2] that
$$\|\Delta_j\big((\Delta_k u)\circ\Phi\big)\|_{L^p}\leq C\min\big(2^{j-k},2^{k-j}\big)\|\nabla\Psi\|_{L^\infty}\|\Delta_k u\|_{L^p}\quad\text{for all}\quad j,k\in\mathbb{Z},$$
so that for $s\in(-1,1)$ and $u\in\dot B^s_{p,r}(\mathbb{R}^3)$, one has
$$\begin{aligned}\|\Delta_j(u\circ\Phi)\|_{L^p}\leq&\sum_{k\in\mathbb{Z}}\|\Delta_j\big((\Delta_k u)\circ\Phi\big)\|_{L^p}\\ \leq&C\Big(\sum_{k\leq j}2^{k-j}+\sum_{k>j}2^{j-k}\Big)\|\nabla\Psi\|_{L^\infty}\|\Delta_k u\|_{L^p}\\ \leq&C\|\nabla\Psi\|_{L^\infty}2^{-js}\Big(\sum_{k\leq j}c_{k,r}2^{(k-j)(1-s)}+\sum_{k>j}c_{k,r}2^{(j-k)(1+s)}\Big)\|u\|_{\dot B^s_{p,r}}\\ \leq&Cc_{j,r}2^{-js}\|\nabla\Psi\|_{L^\infty}\|u\|_{\dot B^s_{p,r}}\quad for\quad(c_{j,r})_{j\in\mathbb{Z}}\in\ell^r(\mathbb{Z}).\end{aligned}$$
This gives
$$\|u\circ\Phi\|_{\dot B^s_{p,r}}\leq C(\|\nabla\Psi\|_{L^\infty})\|u\|_{\dot B^s_{p,r}}\quad for\quad s\in(-1,1).\tag{A.3}$$
Whereas we deduce from (A.2) and (A.3) that
$$\|u\circ\Phi\|_{\dot B^1_{p,r}}\leq\|(\nabla_x u)\circ\Phi\mathcal{B}\|_{\dot B^0_{p,r}}\leq C(\|\nabla\Psi\|_{L^\infty})\big(1+\|\nabla\Psi\|_{\dot B^{\frac3p}_{p,1}}\big)^2\|u\|_{\dot B^1_{p,r}}.$$
For $1<s\leq2$, we get, by using (A.2) and product laws in Besov spaces, that
$$\begin{aligned}\|u\circ\Phi\|_{\dot B^s_{p,r}}\leq&\|(\nabla_x u)\circ\Phi\mathcal{B}\|_{\dot B^{s-1}_{p,r}}\\ \leq&C\big(1+\|\mathcal{B}-I\|_{\dot B^{\frac3p}_{p,1}}\big)\|(\nabla_x u)\circ\Phi\|_{\dot B^{s-1}_{p,r}}\leq C(\|\nabla\Psi\|_{\dot B^{\frac3p}_{p,1}})\|u\|_{\dot B^s_{p,r}}.\end{aligned}$$
This proves the first line of (A.1).

On the other hand, it is easy to observe from Bony's decomposition (3.8) that
$$\|ab\|_{\dot B^\tau_{p,r}}\lesssim\|a\|_{L^\infty}\|b\|_{\dot B^\tau_{p,r}}+\|a\|_{\dot B^{\tau+\frac12}_{p,r}}\|b\|_{\dot B^1_{p,\infty}}\quad\text{for}\quad\tau>0,$$
from which, (A.2), and the first line of (A.1), we infer that for $s\in(2,3]$
$$\begin{aligned}&\|u\circ\Phi\|_{\dot B^s_{p,r}}\lesssim\|(\nabla_x u)\circ\Phi\mathcal{B}\|_{\dot B^{s-1}_{p,r}}\\ &\lesssim(1+\|\nabla\Psi\|_{L^\infty})^2\|(\nabla u)\circ\Phi\|_{\dot B^{s-1}_{p,r}}+(1+\|\nabla\Psi\|_{L^\infty})\|\nabla\Psi\|_{\dot B^{s-\frac12}_{p,r}}\|(\nabla u)\circ\Phi\|_{\dot B^1_{p,r}}\\ &\leq C\big(\|\nabla\Psi\|_{\dot B^{\frac3p}_{p,1}}\big)\big(\|u\|_{\dot B^s_{p,r}}+\|\Psi\|_{\dot B^{s+\frac12}_{p,r}}\|u\|_{\dot B^2_{p,r}}\big).\end{aligned}\tag{A.4}$$
This proves the third line of (A.1).

Inductively we assume that for $k \in \mathbb{N}$ and $k+1 < s-1 \leq k+2$,

$$\|u \circ \Phi\|_{\dot{B}^{s-1}_{p,r}} \leq C\big(\|\nabla \Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\big(\|u\|_{\dot{B}^{s-1}_{p,r}} + \sum_{j=0}^{k-1} \|\Psi\|_{\dot{B}^{s-\frac{1}{2}-j}_{p,r}} \|u\|_{\dot{B}^{j+2}_{p,r}}\big). \tag{A.5}$$

Then by virtue of (A.2) and (A.5), we deduce that

$$\begin{aligned}
\|u \circ \Phi\|_{\dot{B}^{s}_{p,r}} &\lesssim \|(\nabla_x u) \circ \Phi \mathcal{B}\|_{\dot{B}^{s-1}_{p,r}} \\
&\leq C(\|\nabla \Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}})\big(\|(\nabla u) \circ \Phi\|_{\dot{B}^{s-1}_{p,r}} + \|\nabla \Psi\|_{\dot{B}^{s-1}_{p,r}} \|(\nabla u) \circ \Phi\|_{L^\infty}\big) \\
&\leq C(\|\nabla \Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}})\Big(\|u\|_{\dot{B}^{s}_{p,r}} + \sum_{j=0}^{k-1} \|\Psi\|_{\dot{B}^{s-\frac{1}{2}-j}_{p,r}} \|\nabla u\|_{\dot{B}^{j+2}_{p,r}} + \|\Psi\|_{\dot{B}^{s}_{p,r}} \|\nabla u\|_{L^\infty}\Big) \\
&\leq C\big(\|\nabla \Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\big(1 + \|\Delta \Psi\|_{B^{s-2}_{p,r}}\big)\|\nabla u\|_{B^{s-1}_{p,r}},
\end{aligned}$$

which leads to the fifth inequality of (A.1). This concludes the proof of Lemma A.1. □

Appendix B. The proof of Proposition 2.1

The proof of Proposition 2.1 will be based on the following lemma:

Lemma B.1. *Let $s > 2 + \frac{3}{p}$, $p \in (\frac{3}{2}, 2)$, and $f \in B^{s}_{p,1}(\mathbb{R}^3)$ with* Supp $f(x_1, x_2, \cdot) \subset [-K, K]$ *for some positive constant K. We assume moreover that f and $\boldsymbol{b}_0$ are admissible on $\mathbb{R}^2 \times \{0\}$ in the sense of Definition 1.1 and (2.1) holds. Then (1.15) has a solution $\psi \in B^{s}_{p,1}(\mathbb{R}^3)$ so that*

$$\|\psi\|_{B^{s}_{p,1}} \leq C\big(K, \|\nabla \boldsymbol{b}_0\|_{B^{s-1}_{p,1}}\big)\|f\|_{B^{s}_{p,1}}. \tag{B.1}$$

Proof. Due to (2.1), (1.14) has a unique global solution on $\mathbb{R}$ so that for all $t \in \mathbb{R}$,

$$\|\nabla X(t, \cdot)\|_{L^\infty} \leq \exp\Big(\|\nabla \boldsymbol{b}_0\|_{L^\infty}|t|\Big) \quad \text{and} \quad \det\Big(\frac{\partial X(t,x)}{\partial x}\Big) = 1. \tag{B.2}$$

While it follows from (1.15) and (1.14) that

$$\frac{d}{dt}\psi(X(t,x)) = f(X(t,x)),$$

from which, we define

$$\psi(x) = \begin{cases} -\displaystyle\int_0^\infty f(X(t,x))\,dt & \text{if} \quad x_3 \geq 0, \\ \displaystyle\int_{-\infty}^0 f(X(t,x))\,dt & \text{if} \quad x_3 \leq 0. \end{cases}$$

Thanks to the assumption that f and $\boldsymbol{b}_0$ are admissible on $\mathbb{R}^2 \times \{0\}$ in the sense of Definition 1.1, the values of $\psi(x)$ at $(x_1, x_2, 0)$ are compatible. We remark that $b_0^3 \partial_3 \psi = -b_0^1 \partial_1 \psi - b_0^2 \partial_2 \psi + f$ and $b_0^3 \geq \frac{1}{2}$ implies that the derivatives of ψ in the x_1, x_2 variables yields the derivatives of ψ with respect to x_3 variable. Therefore, we do not require any admissible condition for the derivatives of f and $\boldsymbol{b}_0$.

On the other hand, it follows from (2.1) that $b_0^3 \geq \frac{1}{2}$ as long as ε_0 is small enough. So that we deduce from (1.14) that

$$X^3(t,x) \geq x_3 + \frac{t}{2} \geq K \quad \text{if} \quad t \geq 2K, \quad x_3 \geq 0, \qquad \text{and}$$
$$X^3(t,x) \leq x_3 + \frac{t}{2} \leq -K \quad \text{if} \quad t \leq -2K,\ x_3 \leq 0,$$

which together with the assumption: Supp $f(x_1, x_2, \cdot) \subset [-K, K]$ for some positive constant K, implies that

$$\psi(x) = \begin{cases} -\displaystyle\int_0^{2K} f(X(t,x))\,dt & \text{if} \quad x_3 \geq 0, \\ \displaystyle\int_{-2K}^{0} f(X(t,x))\,dt & \text{if} \quad x_3 \leq 0. \end{cases} \tag{B.3}$$

With this solution formula for (1.15), it amounts to prove (B.1) in order to complete the proof of Lemma B.1. Indeed for any $s > 0$, we deduce from (1.14) and product laws in Besov spaces that for any $t \in [-2K, 2K]$

$$\begin{aligned} \|\nabla_x X(t,\cdot) - I\|_{\dot{B}^s_{p,1}} \lesssim \int_0^{|t|} \Big(& \|\nabla \boldsymbol{b}_0\|_{L^\infty} \|\nabla_x X(t',\cdot) - I\|_{\dot{B}^s_{p,1}} \\ & + \|(\nabla \boldsymbol{b}_0)(X(t',\cdot))\|_{\dot{B}^s_{p,1}} \big(1 + \|\nabla_x X(t',\cdot) - I\|_{L^\infty}\big) \Big)\, dt', \end{aligned} \tag{B.4}$$

from which, (A.3) and (B.2), we get, by using Gronwall's inequality, that

$$\max_{t \in [-2K, 2K]} \|\nabla_x X(t,\cdot) - I\|_{\dot{B}^s_{p,1}} \leq C\big(K, \|\nabla \boldsymbol{b}_0\|_{L^\infty}\big) \|\nabla \boldsymbol{b}_0\|_{\dot{B}^s_{p,1}} \quad \text{for} \quad s \in (0,1).$$

Then for $s \in (1,2)$ and $t \in [-2K, 2K]$, we infer

$$\begin{aligned} \|f(X(t,\cdot))\|_{\dot{B}^s_{p,1}} =& \|\nabla f(X(t,\cdot)) \nabla_x X(t,\cdot)\|_{\dot{B}^{s-1}_{p,1}} \\ \lesssim& \|\nabla f\|_{L^\infty} \|\nabla_x X(t,\cdot) - I\|_{\dot{B}^{s-1}_{p,1}} \\ &+ \|\nabla f(X(t,\cdot))\|_{\dot{B}^{s-1}_{p,1}} \big(1 + \|\nabla_x X(t,\cdot) - I\|_{L^\infty}\big) \\ \leq& C\big(K, \|\nabla \boldsymbol{b}_0\|_{L^\infty}\big) \big(\|\nabla f\|_{L^\infty} \|\boldsymbol{b}_0\|_{\dot{B}^s_{p,1}} + \|f\|_{\dot{B}^s_{p,1}}\big). \end{aligned}$$

Notice that for $p \in (\frac{3}{2}, 2)$, $\frac{3}{p} \in (\frac{3}{2}, 2)$, we thus deduce from (B.4) that

$$\begin{aligned} \|\nabla_x X(t,\cdot) - I\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \lesssim& \int_0^{|t|} \|\nabla \boldsymbol{b}_0\|_{L^\infty} \|\nabla_x X(t',\cdot) - I\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\, dt' \\ &+ C\big(K, \|\nabla \boldsymbol{b}_0\|_{L^\infty}\big) \big(\|\nabla^2 \boldsymbol{b}_0\|_{L^\infty} \|\boldsymbol{b}_0\|_{\dot{B}^{\frac{3}{p}}_{p,1}} + \|\nabla \boldsymbol{b}_0\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big), \end{aligned}$$

for $t \in [-2K, 2K]$. Applying Gronwall's inequality gives rise to

$$\max_{t \in [-2K, 2K]} \|\nabla X(t,\cdot) - I\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \leq C\big(K, \|\nabla \boldsymbol{b}_0\|_{B^{1+\frac{3}{p}}_{p,1}}\big) \|\nabla \boldsymbol{b}_0\|_{B^{\frac{3}{p}}_{p,1}}. \tag{B.5}$$

While it is easy to observe from (A.4) that

$$\|u \circ \Phi\|_{\dot{B}^s_{p,1}} \leq C\big(\|\nabla \Psi\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big) \big(\|u\|_{\dot{B}^s_{p,1}} + \|\nabla \Psi\|_{\dot{B}^s_{p,1}} \|u\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)$$

for $s \in (2,3]$, so that for $2 < s-1 \leq 3$, (B.4) implies

$$\|\nabla X(t,\cdot)-I\|_{\dot{B}^{s-1}_{p,1}} \lesssim \int_0^{|t|}\Big(\|\nabla \boldsymbol{b}_0\|_{L^\infty}\|\nabla X(t',\cdot)-I\|_{\dot{B}^{s-1}_{p,1}} + C\big(K,\|\nabla \boldsymbol{b}_0\|_{B^{1+\frac{3}{p}}_{p,1}}\big)\big(\|\nabla \boldsymbol{b}_0\|_{\dot{B}^{s-1}_{p,1}} + \|\nabla X(t',\cdot)-I\|_{\dot{B}^{s-1}_{p,1}}\|\nabla \boldsymbol{b}_0\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\Big)\,dt',$$

for $t \in [-2K,2K]$, applying Gronwall's inequality gives

$$\max_{t\in[-2K,2K]}\|\nabla X(t,\cdot)-I\|_{\dot{B}^{s-1}_{p,1}} \leq C\big(K,\|\nabla \boldsymbol{b}_0\|_{B^{1+\frac{3}{p}}_{p,1}}\big)\|\nabla \boldsymbol{b}_0\|_{\dot{B}^{s-1}_{p,1}} \quad \text{for} \quad s\in(3,4]. \tag{B.6}$$

For $s > 4$, we deduce from (A.1) and (B.4) that

$$\|\nabla X(t,\cdot)-I\|_{\dot{B}^{s-1}_{p,1}} \lesssim \int_0^{|t|}\Big(\|\nabla \boldsymbol{b}_0\|_{L^\infty}\|\nabla X(t',\cdot)-I\|_{\dot{B}^{s-1}_{p,1}} + C\big(K,\|\nabla \boldsymbol{b}_0\|_{B^{1+\frac{3}{p}}_{p,1}}\big)\big(1+\|\nabla X(t',\cdot)-I\|_{B^{s-2}_{p,1}}\big)\|\nabla^2 \boldsymbol{b}_0\|_{B^{s-2}_{p,1}}\Big)\,dt'.$$

Whereas similar to (B.4), one has

$$\|\nabla X(t,\cdot)-I\|_{L^p} \lesssim \int_0^{|t|}\Big(\|\nabla \boldsymbol{b}_0\|_{L^\infty}\|\nabla X(t',\cdot)-I\|_{L^p} + \|\nabla \boldsymbol{b}_0\|_{L^p}\Big)\,dt'.$$

As a consequence, we obtain

$$\|\nabla X(t,\cdot)-I\|_{B^{s-1}_{p,1}} \leq C\big(K,\|\nabla \boldsymbol{b}_0\|_{B^{s-1}_{p,1}}\big)\int_0^{|t|}\Big(\|\nabla X(t',\cdot)-I\|_{B^{s-1}_{p,1}} + \|\nabla \boldsymbol{b}_0\|_{B^{s-1}_{p,1}}\Big)\,dt'.$$

Applying Gronwall's inequality leads to

$$\max_{t\in[-2K,2K]}\|\nabla X(t,\cdot)-I\|_{B^{s-1}_{p,1}} \leq C\big(K,\|\nabla \boldsymbol{b}_0\|_{B^{s-1}_{p,1}}\big)\|\nabla \boldsymbol{b}_0\|_{B^{s-1}_{p,1}} \quad \text{for} \quad s>4. \tag{B.7}$$

Finally we deduce from (A.1) and (B.3) that

$$\|\psi\|_{\dot{B}^s_{p,1}} \leq \int_{-2K}^{2K} C\big(\|\nabla X(t',\cdot)-I\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\big)\Big(1+\|\nabla X(t',\cdot)-I\|_{B^{s-1}_{p,1}}\Big)\|\nabla f\|_{B^{s-1}_{p,1}}\,dt',$$

which together with

$$\|\psi\|_{L^p} \leq \|f\|_{L^p}$$

and (B.6) and (B.7) concludes the proof of (B.1). □

Proof of Proposition 2.1. Under the assumption of (2.1), we would first like to find a solution $(\psi_1,\psi_2) \in B^s_{p,1}(\mathbb{R}^3)$ to the following system:

$$\begin{cases} (1-\partial_{x_2}\psi_2)\partial_{x_3}\psi_1 + \partial_{x_3}\psi_2\partial_{x_2}\psi_1 = b_0^1, \quad \text{for} \quad x\in\mathbb{R}^3,\\ (1-\partial_{x_1}\psi_1)\partial_{x_3}\psi_2 + \partial_{x_3}\psi_1\partial_{x_1}\psi_2 = b_0^2 \end{cases} \tag{B.8}$$

If we use the standard iteration scheme to solve the above problem, the iterated solutions will lose derivative on each step. However, notice that

$$\partial_{x_1}\big(\partial_{x_2}\psi_1\partial_{x_3}\psi_2 + \partial_{x_3}\psi_1(1-\partial_{x_2}\psi_2)\big) + \partial_{x_2}\big(\partial_{x_3}\psi_1\partial_{x_1}\psi_2 + \partial_{x_3}\psi_2(1-\partial_{x_1}\psi_1)\big) + \partial_{x_3}\big((1-\partial_{x_1}\psi_1)(1-\partial_{x_2}\psi_2) - \partial_{x_2}\psi_1\partial_{x_1}\psi_2\big) = 0.$$

This along with $\operatorname{div}\boldsymbol{b}_0 = 0$ ensures that

$$\partial_{x_3}\big(b_0^3 - (1-\partial_{x_1}\psi_1)(1-\partial_{x_2}\psi_2) + \partial_{x_2}\psi_1\partial_{x_1}\psi_2\big) = 0.$$

Since $b_0^3-1\in B^s_{p,1}(\mathbb{R}^3)$, we conclude that

$$b_0^3=(1-\partial_{x_1}\psi_1)(1-\partial_{x_2}\psi_2)-\partial_{x_2}\psi_1\partial_{x_1}\psi_2. \tag{B.9}$$

With (B.9), we solve (B.8) for $\partial_{x_3}\psi_1$ and $\partial_{x_3}\psi_2$ from (B.8)

$$\begin{pmatrix}\partial_{x_3}\psi_1\\ \partial_{x_3}\psi_2\end{pmatrix}=\frac{1}{b_0^3}\begin{pmatrix}1-\partial_{x_1}\psi_1 & -\partial_{x_2}\psi_1\\ -\partial_{x_1}\psi_2 & 1-\partial_{x_2}\psi_2\end{pmatrix}\begin{pmatrix}b_0^1\\ b_0^2\end{pmatrix},$$

or equivalently

$$\begin{aligned}&b_0^1\partial_{x_1}\psi_1+b_0^2\partial_{x_2}\psi_1+b_0^3\partial_{x_3}\psi_1=b_0^1,\\ &b_0^1\partial_{x_1}\psi_2+b_0^2\partial_{x_2}\psi_2+b_0^3\partial_{x_3}\psi_2=b_0^2.\end{aligned} \tag{B.10}$$

Thanks to (2.1) and Lemma B.1, (B.10) has a solution (ψ_1,ψ_2) so that

$$\big\|(\psi_1,\psi_2)\big\|_{B^s_{p,1}}\leq C(K,\varepsilon_0)\big\|(b_0^1,b_0^2)\big\|_{B^s_{p,1}}. \tag{B.11}$$

Whereas for $\mathbf{\Psi}=\big(\psi_1,\psi_2,\psi_3\big)^T$, we deduce from $\det\big(I-\nabla\mathbf{\Psi}\big)=1$ that

$$\begin{aligned}\big(1-\partial_1\psi_1-\partial_2\psi_2+\partial_1\psi_1\partial_2\psi_2-\partial_2\psi_1\partial_1\psi_2\big)\partial_3\psi_3+\big(\partial_2\psi_1\partial_3\psi_2+\partial_3\psi_1(1-\partial_2\psi_2)\big)\partial_1\psi_3&\\ +\big(\partial_3\psi_1\partial_1\psi_2+(1-\partial_1\psi_1)\partial_3\psi_2\big)\partial_2\psi_3=-\partial_1\psi_1-\partial_2\psi_2+\partial_1\psi_1\partial_2\psi_2-\partial_2\psi_1\partial_1\psi_2&,\end{aligned}$$

which together with (B.8) and (B.9) yields

$$b_0^1\partial_{x_1}\psi_3+b_0^2\partial_{x_2}\psi_3+b_0^3\partial_{x_3}\psi_3=b_0^3-1. \tag{B.12}$$

Along the same line to the proof of (B.11), (B.12) has a solution ψ_3 so that

$$\|\psi_3\|_{B^s_{p,1}}\leq C(K,\varepsilon_0)\|b_0^3-1\|_{B^s_{p,1}}.$$

This together with (B.11) leads to (2.2). And thus (2.5) follows from (2.4).

Finally observing that U_0 defined in Proposition 2.1 is in fact the adjoint matrix of $I-\nabla_x\mathbf{\Psi}$, U_0 automatically satisfies (1.9). This finishes the proof of Proposition 2.1. $\square$

Acknowledgments. We would like to thank Professor Fanghua Lin for profitable discussions on this topic. After this work has been finished, Professor Lin told us that he and Ting Zhang proved a similar wellposedness result for a modified MHD system. We also would like to thank Dr. Zhen Lei for pointing out a mistake in the proof of Lemma B.1 in the preliminary version of this paper.

Part of this work was done when the second author was visiting the laboratory of mathematics of Université Paris-Sud in April 2013. He would like to thank the hospitality of the laboratory. Both authors are supported by innovation grant from National Center for Mathematics and Interdisciplinary Sciences. L. Xu is partially supported by NSF of China under Grant 11201455. P. Zhang is partially supported by NSF of China under Grant 10421101 and 10931007.

References

[1] H. Abidi and M. Paicu, Global existence for the magnetohydrodynamic system in critical spaces, *Proc. Roy. Soc. Edinburgh Sect. A*, **138** (2008), 447-476.

[2] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations,* Grundlehren der Mathematischen Wissenschaften, Springer 2011.

[3] J. M. Bony, Calcul symbolique et propagation des singularités pour les ǵquations aux d??rivées partielles non linéaires, *Ann. Sci. École Norm. Sup.*, **14**(4) (1981), 209–246.

[4] H. Cabannes, *Theoretical Magnetofludynamics*, AP, New York, 1970.

[5] C. Cao and J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, *Adv. Math.*, **226** (2011), 1803-1822.
[6] C. Cao, D. Regmi and J. Wu, The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion, *J. Differential Equations*, **254** (2013), 2661-2681.
[7] J. Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Fluids with anisotropic viscosity, *M2AN Math. Model. Numer. Anal.*, **34** (2000), 315-335.
[8] J. Y. Chemin, M. Paicu and P. Zhang, Global large solutions to 3-D inhomogeneous Navier-Stokes system with one slow variable, preprint 2012.
[9] J. Y. Chemin and P. Zhang, On the global wellposedness to the 3-D incompressible anisotropic Navier-Stokes equations, *Comm. Math. Phys.*, **272** (2007), 529–566.
[10] Y. Chen and P. Zhang, The global existence of small solutions to the incompressible viscoelastic fluid System in general space dimensions, *Comm. P. D. E.*, **31** (2006), 1793-1810.
[11] C. Chiuderi and F. Califano, *Phy. Rev. E*, **60** (1999), 4701-4707.
[12] R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, *Invent. Math.*, **141** (2000), 579-614.
[13] T. G. Cowling and D. Phil, *Magnetohydrodynnamics*, the Institute of Physics, 1976.
[14] G. Duvaut and J. L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, *Arch. Ration. Mech. Anal.*, **46** (1972), 241-279.
[15] G. Gui, J. Huang and P. Zhang, Large global solutions to the $3-$D inhomogeneous Navier-Stokes equations, *J. Funct. Anal.* , **261** (2011), 3181-3210;
[16] D. Iftimie, The resolution of the Navier-Stokes equations in anisotropic spaces, *Rev.Mat.Iberoamericana*, **15** (1999), 1–36.
[17] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed. Pergaman, New York, 1984.
[18] Z. Lei, C. Liu and Y. Zhou, Global solutions for incompressible viscoelastic fluids, *Arch. Ration. Mech. Anal.*, **188** (2008), 371-398.
[19] F. Lin, Some analytical issues for elastic complex fluids, *Comm. Pure Appl. Math.*, **65** (2012), 893-919.
[20] F. Lin, C. Liu and P. Zhang, On the hydrodynamics of visco-elasicity, *Comm. Pure Appl. Math.*, **58** (2005), 1437-1471.
[21] F. Lin and P. Zhang, Global small solutions to MHD type system (I): 3-D case, *Comm. Pure Appl. Math.*, (2013) in press.
[22] F. Lin, L. Xu and P. Zhang, Global small solutions to 2-D MHD system, arXiv:1302.5877.
[23] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables,* Applied Mathematical Sciences, 53. Springer-Verlag, New York, 1984.
[24] M. Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, *Revista Matematica Iberoamericana*, **21** (2005), 179–235.
[25] M. Paicu and P. Zhang, Global solutions to the 3-D incompressible anisotropic Navier-Stokes system in the critical spaces, *Comm. Math. Phys.*, **307** (2011), 713-759.
[26] M. Sermange and R. Temam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.*, **36** (1983), 635-664.
[27] L. Xu, P. Zhang and Z. Zhang, Global solvability to a free boundary 3-D incompressible viscoelastic fluid system with surface tension, *Arch. Ration. Mech. Anal.*, **208** (2013), 753-803.
[28] T. Zhang, Erratum to: Global wellposed problem for the 3-D incompressible anisotropic Navier-Stokes equations in an anisotropic space, *Comm. Math. Phys.*, **295** (2010), 877-884.

(L. Xu) LSEC, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, CAS, Beijing 100190, CHINA
E-mail address: `xuliice@lsec.cc.ac.cn`

(P. Zhang) Academy of Mathematics and Systems Science and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, CHINA
E-mail address: `zp@amss.ac.cn`